CR(A) = 0 implies $\mathcal{Z}$-stability

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Abstract

In recent years the study of C*-algebras has seen dramatic results by importing and adapting methods from von Neumann algebras. The importance of various notions of dimension, such as decomposition rank and the nuclear dimension, cannot be understated. Here, we introduce an analogue in the coffee*-setting, the so-called caffeination rank, and take steps towards a converse to a result of Paul Erdos, while extending the class of $\mathcal{Z}$-stable objects well beyond the case of simple nuclear C*-algebras with finite nuclear dimension.

$\mathcal{Z}$-stability

The concept of $\mathcal{Z}$-stability has played an important role in the classification programme for C*-algebras. Introduced by Jiang and Su, $\mathcal{Z}$ was shown to be strongly self-absorbing, terminology introduced by Toms and Winter, meaning that

$$\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z},$$

via an isomorphism approximately unitarily equivalent to $\mathrm{id} \otimes 1_{\mathcal{Z}}$.
Deciding what might be simple unital nuclear C*-algebras are $\mathcal{Z}$-stable has become a major focus of study. Recall that a C*-algebra A is said to be $\mathcal{Z}$-stable if

$$A \cong A \otimes \mathcal{Z}.$$

Interestingly, our recent research suggests that one can determine properties which imply $\mathcal{Z}$-stability for a class strictly larger than the class of simple separable unital nuclear C*-algebras. Here, we extend $\mathcal{Z}$-stability to any A the in the coffee*-class with $CR(A) = 0$.

Going further

Our main motivation for introducing the caffeination rank was as a way of ensuring that mathematician minimise social interactions when caffeination rank is likely to be zero or close to zero. In particular, it seems likely that some people should not be spoken with early in the morning, as they may be unable to communicate properly.

It would be interesting to investigate the role of milk and sugar in coffee. Can the amounts of milk and sugar in coffee be distinguished by the performance of mathematicians? There is some evidence that sugar could be detected, but it is far from clear.

In the long run, we hope that further research will show that mornings should be avoided entirely, and most human activity postponed to early afternoon.

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Background

Mathematics and coffee have enjoyed a long and fruitful relationship of mutual influence. Perhaps the history of these interactions can be traced back to important combinatorics applications to coffee, such as the correspondence between numbers of teaspoons of coffee grounds to the strength (both in flavour as well as caffeine) of the resulting coffee. From the noncommutative perspective, coffee has provided an interesting and somewhat counterintuitive setting. Unlike Connes' noncommutative geometry or the study of C*-algebras stemming from a noncommutative viewpoint of toplogy, the earliest results about coffee were highly noncommutative. Indeed, coffee was consumed in ceramic cups while sitting down, whether at home or in a coffee shop. It wasn't until the much later discovery of travel mugs, takeaway containers, and cupholders in vehicles, that the related commutative theory saw substantial progress.

Perhaps the most well-known result on the interactions between coffee and mathematics is due to Paul Erdos work on the correlation of coffee and number of mathematical theorems proved.

Theorem: [Erdos] A mathematician is a machine for turning coffee into theorems.

A converse was subsequently conjectured, which essentially says that a lack of coffee acts, in some sense, as a forgetful functor, in that a mathematician without coffee loses the ability to think. Below, by "useless", we mean in the sense of mathematical ability.

Conjecture: A mathematician without coffee is useless.

Our work may be seen as a partial converse of Erdos' theorem and an effort towards a solution to the conjecture.

Main Results

Definition: Let A be a mathematician. Then A is said to be in the coffee*-class if A drinks coffee. We let $\mathcal{N}$ denote the coffee*-class.

Remark: Coffee*-mathematicians are generic within the set of all mathematicians.

Definition: Let $A \in \mathcal{N}$ and $d \geq 0$. We say that A has *caffeination rank* at most d if A has injested d standard-size cups of coffee.

Theorem: Let $A \in \mathcal{N}$ be a mathematician and suppose that A has caffeination rank zero. Then A is $\mathcal{Z}$-stable.

Proof: