AN INVITATION TO C^*-ALGEBRAS*

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0. Background and context

The study of C^*-algebras fits into the broader mathematical framework of operator algebras. A phrase one often hears in relation to operator algebras is that they are the "noncommutative" version of classical objects like topological spaces, measure spaces, groups, and so forth. The study of operator algebras started at the beginning of the 20th century. Of course, some noncommutative mathematics, in particular the study of matrices as well as the perhaps more abstract linear operators on finite-dimensional vector spaces, was already known and reasonably developed. Indeed, matrix algebras are themselves (important) examples of operator algebras. However, it is reasonable to say that the subject was brought into being by von Neumann's efforts to make mathematically sound the emerging area of quantum physics. The noncommutative nature that popped up in places such as Heisenberg's uncertainty principal, led von Neumann to define an abstract Hilbert space (particular Hilbert spaces, though they were not called that, were already known, in particular L^2 and ℓ^2 were studied by various people) and undertake a study of operators on such spaces. In particular, he introduced self-adjoint subalgebras of bounded operators that were closed in the weak*-topology; these later became known as von Neumann algebras (the term was coined by Dixmier) and sometimes still as W^*-algebras. Subsequent papers by Murray and von Neumann further developed the subject.

C^*-algebras, which are the topic of this course, have their origins in the studies of Gelfand and Naimark in the 1940's. While it turns out that von Neumann algebras are a particular type of C^*-algebra, they are often studied separately; indeed, there will not be much to say about von Neumann algebras in these notes. Gelfand and Naimark showed that, given a compact Hausdorff space X, the algebra of continuous functions on that space, $C(X)$, can be given an involution as well as norm which makes it a Banach *-algebra. Moreover, this norm satisfies the C^*-equality: $\|f^*f\| = \|f\|^2$ for every $f \in C(X)$. The C^*-equality is precisely what is needed to pass from a Banach *-algebra to a C^*-algebra. This apparently minor requirement has wide-reaching implications for the structure of C^*-algebras; we shall see just what this means in the sequel. Further work by Gelfand and

*or rather, *Another* Invitation to C^*-algebras, for those who either missed or turned down Arveson's 1998 invitation.

Naimark, together with Segal, established way of constructing a representation of any C*-algebra as a norm-closed self-adjoint subalgebra of bounded operators on a Hilbert space. Gelfand and Naimark showed that in fact there is always a *faithful* representation; thus every C*-algebra is isomorphic to such a subalgebra of operators.

Since those early days, the study of C*-algebras has taken off in many directions. It continues to find utility in the study of physics (including quantum gravity, quantum information, statistical mechanics), the study of topological groups as well as the development of quantum groups, dynamical systems, K-theory, and of course, noncommutative geometry. Some of the state-of-the-art will have been introduced at the master classes. Here, we concentrate on developing the theory from the beginning, with minimal background requirements save for some knowledge of functional analysis and Hilbert spaces.

Contents

1. Banach algebras and spectral theory

A *Banach algebra* is an algebra A, together with a submultiplicative norm $\|\cdot\| : A \to [0,\infty)$, which is complete with respect to the norm. For the purposes of this course, we will consider only algebras over $\mathbb{C}$.

Note that A is not necessarily unital. If A has a unit 1_A then we require that $\|1_A\| = 1$ and in this case we call A a *unital* Banach algebra.

If $B \subset A$ is a subalgebra, then its closure with respect to the norm of A is also a Banach algebra.

All Banach we will consider will be $\mathbb{C}$-algebras, though they can also be defined over other fields.

1.1 Examples: (a) Let X be a topological space and let

$$C_b(X) := \{f : X \to \mathbb{C} \mid f \text{ continuous, bounded }\}.$$

Then $C_b(X)$ is a Banach algebra when equipped with pointwise operations and supremum norm

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in X.\}$$

(b) Let X be a Banach space. Let $\mathcal{L}(X) = \{T : X \to X \mid T \text{ linear, continuous }\}$ equipped with pointwise addition and composition for multiplication. $\mathcal{L}(X)$ is a Banach algebra with the operator norm

$$\|T\| = \sup_{x \in X, \|x\| \leq 1} \|T(x)\|.$$

(c) Let (X, Σ, μ) be a measure space. Let

$$L^\infty(X, \Sigma, \mu) := \{f : X \to \mathbb{C} \mid f \text{ measurable, and } \exists K > 0 \text{ s.t. } \mu(\{x \mid |f(x)| > K\}) = 0\}.$$

Define a norm on $L^\infty(X, \Sigma, \mu)$ by

$$\|f\| = \inf_{f = g \; a.e. \mu} \sup_{x \in X} |g(x)|.$$

Then $L^\infty(X, \Sigma, \mu)$ is a Banach algebra.

Spectrum. Let $p(z) = \lambda_0 + \lambda_1 z + \ldots \lambda_n z^n$, $\lambda_i \in \mathbb{C}$, be a polynomial in the algebra of polynomials in one indeterminate, $\mathbb{C}[z]$. Let A be a unital algebra and $a \in A$ and denote by $p(a)$ the element $\lambda_0 1_A + \lambda_1 a + \ldots \lambda_n a^n$.

Let A be a unital algebra. An element $a \in A$ is invertible if there is a $b \in A$ such that $ab = ba = 1_A$. In this case we write $b = a^{-1}$. (This makes sense because if such a b exists, it is is unique. Why?)

1.2 Exercise: Let A be a unital algebra and show that the set of invertible elements $\text{Inv}(A) := \{a \in A \mid a \text{ is invertible in } A\}$ is a group under multiplication.

The spectrum of an element a in the unital algebra A is defined to be

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{C} \mid \lambda \cdot 1_A - a \notin \mathrm{Inv}(A)\}.$$

1.3 Exercise: What is $\mathrm{sp}(a)$ for $a \in M_n$ and $\mathrm{sp}(f)$ for $f \in C(X)$, where X is a compact Hausdorff space?

1.4 Suppose that $1 - ab$ is invertible with inverse c. Then one can check that $1 - ba$ is also invertible with inverse given by $1 + bca$. As a result, we have that, for any a, b in a unital Banach algebra A,

$$\mathrm{sp}(ab) \setminus \{0\} = \mathrm{sp}(ba) \setminus \{0\}.$$

We have the following spectral mapping property for polynomials.

1.5 Theorem: *Let A be a unital algebra, $a \in A$ and p a polynomial in $\mathbb{C}[z]$. Suppose that $\mathrm{sp}(a) \neq \emptyset$. Then $\mathrm{sp}(p(a)) = p(\mathrm{sp}(a))$.*

Proof: If p is constant, the result is obvious, so we may assume otherwise. Let $\mu \in \mathbb{C}$ and consider the polynomial $p - \mu$. Since every polynomial over $\mathbb{C}$ splits, we can write

$$p(z) - \mu = \lambda_0(\lambda_1 - z) \cdots (\lambda_n - z)$$

for some $n \in \mathbb{N} \setminus \{0\}$, $\lambda_0, \dots, \lambda_n \in \mathbb{C}$ and $\lambda_0 \neq 0$. If $\mu \notin \mathrm{sp}(p(a))$ then $p(a) - \mu$ is invertible and hence each $\lambda_i - a$, $i = 1, \dots, n$, is invertible. Conversely, it is clear that if each $\lambda_i - a$ is invertible, so is $p(a) - \mu$. Thus $\mu \in \mathrm{sp}(p(a))$ if and only if there is some $1 \leq i \leq n$ with $\lambda_i \in \mathrm{sp}(a)$ and we have $\mathrm{sp}(p(a)) \subseteq p(\mathrm{sp}(a))$. Now if $\lambda \in \mathrm{sp}(a)$ then $p(a) - p(\lambda) = (\lambda - a)b$ for some $b \in A$ and hence is not invertible. Thus $\mathrm{sp}(p(a)) = p(\mathrm{sp}(a))$. ∎

1.6 Exercise: Let $\mathbb{C}(z)$ denote the field of fractions of $\mathbb{C}[z]$. Show that there is an element in $\mathbb{C}(z)$ which has empty spectrum.

The above shows that for a general unital algebra, it is possible for an element to have empty spectrum. In a unital Banach algebra, however, this is not the case. This is Theorem 1.11 below. To prove it, we require a few preliminary results. The first is that in a unital Banach algebra where we have a notion of convergence, we have a theorem for a geometric series which will prove quite useful in the sequel.

1.7 Theorem: *Let A be a unital Banach algebra and $a \in A$ such that $\|a\| < 1$. Then $1 - a$ is invertible and*

$$(1 - a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

PROOF: First we note that by submultiplicativity of the norm together with the usual convergence of a geometric series, we have that

$$\left\|\sum_{n=0}^{\infty} a^n\right\| \leq \sum_{n=0}^{\infty} \|a\|^n$$

is finite. Since A is complete, this means that $\sum_{n=0}^{\infty} a^n$ converges to some $b \in A$. Since

$$(1-a)\sum_{n=0}^{N} a^n = \sum_{n=0}^{N} a^n - \sum_{m=1}^{N+1} a^m \to 1, N \to \infty,$$

we must have $b = (1-a)^{-1}$, as claimed. ▮

1.8 LEMMA: *Let A be a unital Banach algebra. Then* $\mathrm{Inv}(A)$ *is open in A and a the map*

$$\mathrm{Inv}(A) \to \mathrm{Inv}(A) : \quad a \mapsto a^{-1}$$

is differentiable.

PROOF: Let $a \in \mathrm{Inv}(A)$. We will show that any b sufficiently close to a is also invertible, which will show the first part of the lemma. Let $b \in A$ such that $\|a-b\|\|a^{-1}\| < 1$. Then

$$\|ba^{-1} - 1\| \leq \|a^{-1}\|\|a-b\| < 1,$$

so, by the previous theorem we have that ba^{-1} is invertible. Thus we also have that $b(a^{-1}(ba^{-1})^{-1}) = 1$, so b is invertible.

To show that $a \mapsto a^{-1}$ is differentiable, we need to find a linear map $L : A \to A$ such that, for $a \in \mathrm{Inv}(A)$,

$$\lim_{h\to 0} \frac{\|(a+h)^{-1} - a^{-1} - L(h)\|}{\|h\|} = 0.$$

Define, for $b \in A$, $L(b) = -a^{-1}ba^{-1}$.

Let $a \in A$ be invertible and let h be small enough that

$$\|h\|\|a^{-1}\| < 1/2.$$

Then we have that $\|a^{-1}h\| < 1/2$ so, by Theorem 1.7 $1 + a^{-1}h$ is invertible and

$$\begin{aligned}
\|(1+a^{-1}h)^{-1} - 1 + a^{-1}h\| &= \|\textstyle\sum_{n=0}^{\infty}(-1)^n(a^{-1}h)^n - 1 + a^{-1}h\| \\
&= \|\textstyle\sum_{n=2}^{\infty}(-1)^n(a^{-1}h)^n\| \\
&\leq \sum_{n=2}^{\infty} \|(a^{-1}h)\|^n \\
&\leq \|a^{-1}h\|^2/(1 - \|a^{-1}h\|)^{-1} \\
&\leq 2\|a^{-1}h\|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\|(a+h)^{-1}-a^{-1}-L(h)\|}{\|h\|} &= \frac{\|(a+h)^{-1}-a^{-1}+a^{-1}ha^{-1})\|}{\|h\|} \\
&= = \frac{\|(a(a^{-1}a+a^{-1}h))^{-1}-(1-a^{-1}h)a^{-1}\|}{\|h\|} \\
&\leq \frac{\|(1+a^{-1}h)^{-1}-1+a^{-1}h\|\|a^{-1}\|}{\|h\|} \\
&\leq \frac{2\|a^{-1}\|^2\|h\|^2}{\|h\|},
\end{aligned}$$

which goes to zero as h goes to zero. ▮

In a metric space X, we will denote the open ball of radius $r > 0$ about a point $x \in X$ by $B(x,r)$ and its closure by $\overline{B(x,r)}$.

1.9 Lemma: *Let A be a unital Banach algebra. Then for any $a \in A$, the spectrum of a is a closed subset of $\overline{B(0,\|a\|)} \subset \mathbb{C}$ and the map*

$$\mathbb{C} \setminus \mathrm{sp}(a) \to A : \lambda \mapsto (a-\lambda)^{-1}$$

is differentiable.

Proof: Once we show that $\mathrm{sp}(a) \subset \overline{B(0,\|a\|)}$, the rest follows from the previous lemma. To prove $\mathrm{sp}(a) \subset \overline{B(0,\|a\|)}$, we need to show that if $|\lambda| > \|a\|$ then $\lambda - a$ is invertible. The details are left as an exercise. ▮

1.10 Now we are able to prove that every element in a unital Banach algebra has nonempty spectrum. In what follows, A^* denotes the dual space of A, that is

$$A^* = \{f : A \to \mathbb{C} \mid f \text{ continuous and linear.}\}$$

A^* can be given a topology called the *weak*-topology*, which is the topology generated by semi-norms of the form $p_a(\tau) = |\tau(a)|$ ranging over all $a \in A$. A sequence $(\phi_n)_{n\in\mathbb{N}}$ converges to $\phi \in A^*$ if $\phi_n(a) \to \phi(a), n \to \infty$ for every $a \in A$ (pointwise convergence). For further details see, for example, [1, Appendix].

1.11 Theorem: *Let A be a unital Banach algbera. Then for every $a \in A$ we have $\mathrm{sp}(a) \neq \emptyset$.*

Proof: First of all, we may assume that a is nonzero since $0 \in \mathrm{sp}(0)$. So let $a \neq 0$ and assume, for contradiction, that $\mathrm{sp}(a) = \emptyset$. We leave it as an exercise to show that the map

$$\mathbb{C} \to \mathrm{Inv}(A) : \lambda \mapsto (a-\lambda)^{-1}$$

is bounded on the compact disc of radius $2\|a\|$. Once this has been shown it follows that for any $\phi \in A^*$ the map

$$\lambda \mapsto \phi((a-\lambda)^{-1})$$

is also bounded. From the previous theorem, this map is also entire, which, by Liouville's theorem, implies it must be constant. Thus $\phi(a^{-1}) = \phi((a-1)^{-1})$ for every $\phi \in A^*$ leading to the contradiction that $a^{-1} = (a-1)^{-1}$. ▮

The following is an immediate consequence.

1.12 THEOREM: *Let A be a unital Banach algebra with* $\mathrm{Inv}(A) = A \setminus \{0\}$. *Then* $A = \mathbb{C}$.

1.13 The *spectral radius* of an element a in a unital Banach algebra A is defined to be

$$r(a) := \sup_{\lambda \in \mathrm{sp}(a)} |\lambda|.$$

We have the following characterisation of the spectral radius which relates it to the norm of the element a.

THEOREM: *For any $a \in A$ we have* $r(a) = \inf_{n \geq 1} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^n\|^{1/n}$.

PROOF: Since $\lambda \in \mathrm{sp}(a)$ implies $\lambda^n \in \mathrm{sp}(a^n)$ we have $|\lambda^n| \leq \|a^n\|$. Thus $|\lambda| = |\lambda^n|^{1/n} \leq \|a^n\|^{1/n}$ for every $\lambda \in \mathrm{sp}(a)$ and every $n \geq 1$, that is,

$$r(a) = \sup_{\lambda \in \mathrm{sp}} |\lambda| \leq \inf_{n \geq 1} \|a^n\|^{1/n}.$$

By definition we have that $\inf_{n\geq 1} \|a^n\|^{1/n} \leq \liminf_{n\to\infty} \|a^n\|^{1/n}$, thus we are finished if we show that $r(a) \geq \limsup_{n\to\infty} \|a^n\|^{1/n}$.

Let $D = B(0, 1/r(a))$ if $r(a) \neq 0$ and $D = \mathbb{C}$ otherwise. If $\lambda \in D$ then $1 - \lambda a$ is invertible by Theorem 1.7. It follow from Lemma 1.9 that, for every $\phi \in A^*$ the map

$$f : D \to \mathbb{C} : \lambda \mapsto \phi((1 - \lambda a)^{-1})$$

is analytic. Thus there are unique complex numbers $(\lambda_n)_{n\in\mathbb{N}}$ such that

$$f(\lambda) = \sum_{n=0}^{\infty} \lambda_n \lambda^n$$

whenever $\lambda \in D$.

But again, by applying Theorem 1.7, we have, for $\lambda < 1/\|a\| \leq 1/r(a)$

$$(1 - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n.$$

It follows that $f(\lambda) = \sum_{n=0}^{\infty} \lambda^n \phi(a^n)$, so that $\phi(a^n) = \lambda_n$ for every $n \in \mathbb{N}$.Thus $\phi(a^n) \to 0$ as $n \to \infty$ and therefore the sequence $(\phi(a^n))_{n\in\mathbb{N}}$ is bounded. This is true for every $\phi \in A^*$ so in fact $(\|\lambda^n a^n\|)_{n\in\mathbb{N}}$ is also bounded by some $M_\lambda > 0$. Thus

$$\|a^n\|^{1/n} \leq M_\lambda^{1/n}/|\lambda|,$$

so

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \leq 1/|\lambda|$$

for every $\lambda \in D$. It follows that

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \leq r(a)$$

as required. ∎

1.14 Exercise: Let A be a (not necessarily unital) Banach algebra. Let $\tilde{A} := A \oplus \mathbb{C}$ as a vector space. Define a multiplication by

$$(a, \lambda) \cdot (b, \mu) = (ab + \lambda b + \mu a, \lambda\mu),$$

and a norm by

$$\|(a, \lambda)\| = \|a\| + |\lambda|$$

Show that $\tilde{A}$ is a unital Banach algebra.

Remark: When A is non unital $\tilde{A}$ is called the *unitisation* of A. When we consider C*-algebras in Section 3, we will have to be a little bit more careful in defining the norm.

1.15 Exercise: Let A be a nonunital Banach algebra. The spectrum of $a \in A$ is defined to be the spectrum of a in $\tilde{A}$, that is

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{C} \mid \lambda \cdot 1_{\tilde{A}} - a \notin \mathrm{Inv}(\tilde{A})\}.$$

Give a one-line proof of Theorem 1.11 (without using Theorem 1.11!) in the case that A is nonunital.

Exercises

1.1 Check that Examples 1.1 give Banach algebras.

1.2 Let $\mathbb{C}[z]$ denote the single-variable $\mathbb{C}$-valued polynomials, equipped with pointwise operations and norm $\|p\| = \sup_{|z|=1} |p(z)|$. Is this a Banach algebra?

1.3 Let A be a Banach algebra. Show that multiplication in A is continuous.

1.4 Let H be a Hilbert space with orthonormal basis $(e_i)_{i\in I}$. An operator $T \in \mathcal{B}(H)$ is a *Hilbert–Schmidt operator* if $\sum_{i\in I} \|Te_i\|^2$ is finite. The Hilbert–Schmidt norm $\|T\| = (\sum_{i\in I} \|Te_i\|^2)^{1/2}$ can be defined on the set of all Hilbert–Schmidt operators. With the usual operations for operators on a Hilbert space, are the Hilbert–Schmidt operators a Banach algebra?

1.5 Let A be a unital algebra and show that the set of invertible elements $\mathrm{Inv}(A) := \{a \in A \mid a \text{ is invertible in } A\}$ is a group under multiplication.

1.6 What is $\mathrm{sp}(a)$ for $a \in M_n$ and $\mathrm{sp}(f)$ for $f \in C(X)$, where X is a compact Hausdorff space?

1.7 Let $\mathcal{H} = \ell^2(\mathbb{Z}) = \{(\lambda_n)_{n\in\mathbb{Z}} \mid \sum_{|n|=0}^{\infty} |\lambda_n|^2 \text{ converges}\}$. Define the bilateral shift operator $S : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by

$$S((\lambda_n)_{n\in\mathbb{Z}}) = (\mu_n)_{n\in\mathbb{Z}}$$

where $\mu_n = \lambda_{n-1}$.
(a) Show that $S \in \mathbb{B}(\mathcal{H})$
(b) What is S^*? Is S invertible? If so, what is its inverse?
(c) Show that S has no eigenvalues (i.e. for every $\lambda \in \mathbb{C}$ there is no $\xi \in \mathcal{H}$ such that $S\xi = \lambda \cdot \xi$.) Hint: If $S\xi = \lambda \cdot \xi$ is $S\xi \in \ell^2$?
(d) Show that if $|\lambda| = 1$ then $\lambda \cdot 1_{\mathbb{B}(\mathcal{H})} - S$ is not invertible.

1.8 Let $H = L^2([0,1]) = \{f : [0,1] \to \mathbb{C} \mid \int f^2 < \infty\}$, and consider the Banach algebra $\mathcal{B}(H)$ (b) Let $T \in \mathcal{B}(H)$ be defined as

$$T(f)(t) = \int_0^t f(x)dx.$$

Compute the spectral radius of T. What is $\mathrm{sp}(T)$?

1.9 Let X be a compact space and A a unital Banach algebra. Show that

$$C(X, A) := \{f : X \to A \mid f \text{ continuous}\}$$

can be given the structure of a Banach algebra. In the case that $A = M_n$ we have that $C(X, A) \cong M_n(A)$.

1.10 Let $\mathbb{C}(z)$ denote the field of fractions of $\mathbb{C}[z]$. Show that there is an element in $\mathbb{C}(z)$ which has empty spectrum.

1.11 Let A be a unital Banach algebra and $a \in A$. Show that $\mathrm{sp}(a) \subset \overline{B(0, \|a\|)}$ (show that if $|\lambda| > \|a\|$ then $\lambda - a$ is invertible).

1.12 Show that the map in Theorem 1.11,

$$\mathbb{C} \to \mathrm{Inv}(A) : \lambda \mapsto (a - \lambda)^{-1}$$

is bounded on the compact disc of radius $2\|a\|$.

1.13 Let A be a (not necessarily unital) Banach algebra. Let $\tilde{A} := A \oplus \mathbb{C}$ as a vector space. Define a multiplication by

$$(a, \lambda) \cdot (b, \mu) = (ab + \lambda b + \mu a, \lambda\mu),$$

and a norm by

$$\|(a, \lambda)\| = \|a\| + |\lambda|$$

Show that $\tilde{A}$ is a unital Banach algebra.

1.14 Let A be a nonunital Banach algebra. The spectrum of $a \in A$ is defined to be the spectrum of a in $\tilde{A}$, that is

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{C} \mid \lambda \cdot 1_{\tilde{A}} - a \notin \mathrm{Inv}(\tilde{A})\}.$$

Give a one-line proof of Theorem 1.11 (without using Theorem 1.11!) in the case that A is nonunital.

1.15 Let A be a untial Banach algebra and $B \subset A$ with $1_A \in B$.

(a) Show that $\mathrm{Inv}(B)$ is a clopen subset of $\mathrm{Inv}(A) \cap B$.

(b) Let $b \in B$. Show that $\mathrm{sp}_A(b) \subset \mathrm{sp}_B(b)$ and $\partial\,\mathrm{sp}_B(b) \subset \partial\,\mathrm{sp}_A(b)$. Show that if $\mathbb{C} \setminus \mathrm{sp}_A(b)$ has exactly one bounded component ($\mathrm{sp}_A(b)$ has no holes), then $\mathrm{sp}_A(b) = \mathrm{sp}_B(b)$.

2. The Gelfand representation

2.1 Let A be an algebra. A subset $I \subset A$ is a right (left) ideal if $a \in A$ and $b \in B$ then $ab \in B$ ($ba \in B$). We will call $I \subset A$ an ideal if it is both a right and a left ideal. When I is an ideal, then A/I is also an algebra with the obvious definitions for multiplication and addition.

A/I is a unital algebra exactly when I is a *modular* ideal: there exists and element $u \in A$ such that $a - ua \in I$ and $a - au \in I$ for every $a \in A$. (What is $1_{A/I}$?). Note that this implies that every ideal in a unital algebra is modular.

If A is a Banach algebra and the ideal I is norm-closed then A/I can be given the quotient norm

$$\|a + I\| = \inf_{b \in I} \|a + b\|, \quad a \in A$$

making A/I into a Banach algebra.

2.2 We also have the usual notions of trivial ideals ($= 0, A$) and ideals generated by a set $J \subset A$ (= smallest ideal containing J). A proper ideal is one which is not equal to A (but may be zero) and a maximal ideal is a proper ideal not contained in any other proper ideal. One can use a Zorn's Lemma argument to show that every proper modular ideal is contained in a maximal modular ideal. In particular, if A is unital then A every proper ideal of A is contained in a maximal ideal.

2.3 Exercise: If A is a Banach algebra then it is a proper closed maximal ideal in its unitisation $\tilde{A}$ (as defined in Exercise 1.14).

2.4 Proposition: *Let A be a Banach algebra and $I \subset A$ an ideal. If I is proper and modular, than $\overline{I}$ is also proper.*

Proof: Since I is modular, there is an element $u \in A$ such that $a - ua \in I$ and $a - au \in I$ for every $a \in A$. Let $b \in I$ with $\|u - b\| < 1$. Then $1 - u + b$ is invertible as an element of $\tilde{A}$. Let c denote its inverse. Then

$$1 = c(1 - u + b) = c - cu + cb \in I,$$

contradicting the fact that I is proper. Thus any $b \in I$ must satisfy $\|u - b\| \geq 1$. In particular, $u \in A \setminus \overline{I}$, so $\overline{I}$ is proper. ∎

2.5 Corollary: *If I is a maximal modular ideal then it is closed.*

2.6 Proposition: *Let A be a unital commutative algebra and $I \subset A$ a modular ideal. If A is maximal then A/I is a field.*

Proof: Exercise. ∎

2.7 If A and B are Banach algebras, a map $\phi : A \to B$ is a homomorphism if it is an algebra homomorphism that is continuous with respect to the norms of A and B. If A and B are unital and $\phi(1_A) = 1_B$ then we call ϕ a unital homomorphism. The norm of a given homomorphism $\phi : A \to B$ is defined to be

$$\|\phi\| = \sup\{\|\phi(a)\|_B \mid a \in A, \|a\|_A \leq 1\}.$$

2.8 Exercise: Let $\phi : A \to B$ be a homomorphism of Banach algebras A and B. Show that $\ker(\phi)$ is a closed ideal in A.

2.9 Recall from Exercise 1.14 that $\tilde{A}$ denotes the unitisation of nonunital Banach algebra A. The map $\iota : A \to \tilde{A}$ given by $\iota(a) = (a, 0)$ is an injective homomorphism, so we may identify A as a subalgebra in $\tilde{A}$. We also have the canonical projection map $\pi : \tilde{A} \to \mathbb{C}$ given by $\pi((a, \lambda)) = \lambda$. Its kernel is clearly A, so A is in fact an ideal in $\tilde{A}$.

2.10 Definition: Let A be a Banach algebra. A *character on A* is a nonzero homomorphism $\tau : A \to \mathbb{C}$. Let

$$\Omega(A) := \{\tau : A \to \mathbb{C} \mid \tau \text{ a character on } A\}$$

We call $\Omega(A)$ the character space of A, or based on what we'll see below, the spectrum of A.

For commutative Banach algebras there is an important relation between characters, maximal ideals, Banach algebras of the form $C_0(X)$ for some locally compact Hausdorff spaces X.

2.11 Theorem: *Let A be a unital commutative Banach algebra. Then*

(i) $\tau(a) \in \mathrm{sp}(a)$ *for every* $\tau \in \Omega(A)$ *and every* $a \in A$,
(ii) $\|\tau\| = 1$, *and*
(iii) $\Omega(A) \neq \emptyset$ *and* $\tau \to \ker\tau$ *is a bijection from* $\Omega(A)$ *to the set of maximal ideals of* A.

Proof: The proof is left as an exercise. For a hint, if I is a maximal ideal use Theorem 2.6 to define a homomorphism from $A \to \mathbb{C}$. ∎

Note that (ii) above says that $\Omega(A)$ is contained in the closed unit ball of the dual space A^*. Thus we may endow $\Omega(A)$ with the weak-* topology inherited from A^*.

2.12 Theorem: *Let A be a unital commutative Banach algebra. Then, for any $a \in A$*

$$\mathrm{sp}(a) = \{\tau(a) \mid \tau \in \Omega(A)\}.$$

Proof: Suppose that $\lambda \in \mathrm{sp}(a)$. The ideal generated by $(a - \lambda)$ is proper since it can't contain 1_A. It is therefore contained in some maximal ideal which is of the form $\ker(\tau)$ for some $\tau \in \Omega(A)$, in which case $\tau(a) = \lambda$. Conversely, $\tau(\tau(a) - a) = 0$, so $\tau(a) \in \mathrm{sp}(a)$. ∎

The proof of this next corollary is a relatively easy exercise:

2.13 Corollary: *Let A be a nonunital commutative Banach algebra. Then, for any $a \in A$,*

$$\mathrm{sp}(a) = \{\tau(a) \mid \tau \in \Omega(A)\} \cup \{0\}.$$

2.14 Theorem: *Let A be a commutative Banach algebra. Then $\Omega(A)$ is a locally compact Hausdorff space. If A is unital, then $\Omega(A)$ is compact.*

Proof: It follows from Theorem 2.11 that $\Omega(A) \setminus \{0\}$ is a weak-* subset of the closed unit ball A^*. Thus by the Banach–Alaoglu theorem, it is compact. Hence $\Omega(A)$ is locally compact. If A is unital then one checks that in fact $\Omega(A)$ itself is closed, hence compact. ∎

We will now show the existence of the Gelfand representation, which says that we can represent any commutative Banach algebra A as functions on a locally compact Hausdorff space which is homeomorphic to $\Omega(A)$. When we move on to the next section, we will see that this has particularly nice consequences for C*-algebras, in particular, it will give us a continuous functional calculus which is an indispensable tool to the theory. But first, we remain in the more general world of Banach algebras.

2.15 Let $a \in A$ and define $\hat{a} : \Omega(A) \to \mathbb{C}$ by $\hat{a}(\tau) = \tau(a)$. Then $\hat{a} \in C_0(\Omega(A))$ (indeed, the weak-* topology is the coarsest topology making every $\hat{a}$, $A \in A$ continuous; this can be taken as its definition). The map $a \to \hat{a}$ is called the *Gelfand transform* and $\hat{a}$ is the *Gelfand transform* of a.

2.16 Theorem: *Let A be a commutative Banach algebra with $\Omega(A) \neq \emptyset$. Then*

$$A \to C_0(\Omega(A)) : a \mapsto \hat{a}$$

is a norm-decreasing homomorphism and, moreover

$$r(a) = \|\hat{a}\|.$$

if A is unital and $a \in A$ then $\mathrm{sp}(a) = \hat{a}(\Omega(A))$. *When A is nonunital and $a \in A$, then* $\mathrm{sp}(a) = \hat{a}(\Omega(A)) \cup \{0\}$.

Proof: By Theorem 2.12 and Corollary 2.13 we have $r(a) = \|\hat{a}\|$. Since $r(a) \leq \|a\|$, the map is norm-decreasing. It is easy to check that it is also a homomorphism. ▮

2.17 Theorem: *Let A be a unital Banach algebra and let $a \in A$. Let $B \subset A$ be the Banach algebra generated by a and 1_A. Then B is commutative and the map*

$$\hat{a} : \Omega(B) \to \mathrm{sp}(a)$$

defined by

$$\hat{a}(\phi) = \phi(a).$$

is a homeomorphism.

Proof: It is clear that B is commutative and since $\hat{a}$ is a continuous bijection and $\Omega(B)$ is compact, it is a homeomorphism. ▮

Exercises

2.1 By a homomorphism of Banach algebras we mean a continuous algebra homomorphism. Let $\phi : A \to B$ be a homomorphism of Banach algebras A and B. Show that $\ker(\phi)$ is a closed ideal in A.

2.2 Let I be a (not necessarily closed) ideal in a unital Banach algebra A. Show that if I is maximal, $I = \overline{I}$.

2.3 An ideal I in a (not necessarily unital) Banach algebra A is a *modular* ideal if there exists and element $u \in A$ such that $a - ua \in I$ and $a - au \in I$ for every $a \in A$.

(a) Show that every proper modular ideal is contained in a maximal modular ideal.

(b) Let I be a maximal modular ideal. Show that A/I is a field.

2.4 Show that A is a maximal ideal in its unitisation $\tilde{A}$.

2.5 Prove Corollary 2.13: If A is a nonunital Banach algebra and $a \in A$ then

$$\mathrm{sp}(a) = \{\tau(a) \mid \tau \in \Omega(A)\} \cup \{0\}.$$

3. C*-algebra basics

3.1 A *-algebra is a $\mathbb{C}$-algebra A together with an involutive (that is, an antilinear order two isomorphism) map $^* : A \to A$. Given an element a in a *-algebra A, we call a^* the *adjoint* of a. An element in A is called *self-adjoint* if $a = a^*$. An element $p \in A$ is called a *projection* if it is self-adjoint and $p^2 = p$. An element $a \in A$ which commutes with its adjoint is called *normal*. When A is unital and $u \in A$ is a normal element such that $u^*u = 1$ then we call u a *unitary*.

We will denote the set of self-adjoint elements in A by A_{sa} and the unitaries by $\mathcal{U}(A)$.

3.2 Exercise: Let A be a *-algebra. What is $\mathrm{sp}(a^*)$?

3.3 If A is a Banach algebra with involution *, then we call A a Banach *-algebra if $\|a^*\| = \|a\|$ for every $a \in A$.

3.4 An *abstract* C**-algebra* is Banach *-algebra whose norm satisfies the following equality, called the C**-equality*

$$\|a^*a\| = \|a\|^2 \text{ for every } a \in A.$$

We often call a norm which satisfies the C*-equality a C**-norm*.

What may appear to be only a minor requirement for the norm in fact gives a C*-algebra many nice structural properties that we don't see in an arbitrary Banach algebra. First, let's look at a few examples.

3.5 Example: Let $n \in \mathbb{N}$ and $M_n := M_n(\mathbb{C})$ be the set of $n \times n$ matrices with complex entries. Equip M_n with the operator norm, that is, $\|A\| = \sup_{x\in\mathbb{C}^n, \|x\|=1} |A(x)|$. Then M_n is a C*-algebra under the usual matrix multiplication and addition and with involution given by taking adjoints.

More generally, if H is a Hilbert space then $\mathcal{B}(H)$ is a C*-algebra.

3.6 Example: Let H be a Hilbert space and let $\mathcal{K}(H)$ denote the subalgebra of compact operators. Then $\mathcal{K}(H)$ is also a C*-algebra with the inherited structure from $\mathcal{B}(H)$.

More generally, if A is any closed self-adjoint subalgebra of $\mathcal{B}(H)$, then A is also a C*-algebra with the inherited structure. Such a C*-algebra is called a *concrete* C*-algebra.

3.7 Example: Let X be a locally compact Hausdorff space. We say that a function $f : X \to \mathbb{C}$ *vanishes at infinity* if, for every $\epsilon > 0$ there is a compact set $K \subset X$ such that $|f(x)| < \epsilon$ for every $x \in X \setminus K$. Let $C_0(X) = \{f : X \to \mathbb{C} \mid f \text{ vanishes at infinity}\}$, and equip $C_0(X)$ with pointwise operations, sup norm and for $f \in C_0(X)$, define $f^*(x) := \overline{f(x)}$. Then $C_0(X)$ is a C*-algebra which is unital if and only if X is compact. In this case we denote $C_0(X)$ by $C(X)$.

3.8 Example: By an ideal I in a C*-algebra A, we will mean a self-adjoint, closed two-sided ideal. Given such an ideal, we may define the quotient C*-algebra A/I as we did for a Banach algebra together with the obvious involution induced from A.

A few nice properties that we obtain from having a C*-norm can be noticed straight away. For example, it is automatic in a unital C*-algebra we have $\|1\| = 1$. More generally, if u is a unitary, $\|u\| = 1$ and also if p is a projection then $\|p\| = 1$.

3.9 This gives us information about the spectrum of a unitary u: Using Lemma 1.9, if $\lambda \in \mathrm{sp}(u)$ then $|\lambda| \leq \|u\| = 1$. Since u is invertible, we must also have $\lambda^{-1} \in \mathrm{sp}(u^{-1}) = \mathrm{sp}(u^*) \leq \|u^*\| = 1$. Thus $|\lambda| = 1$ so $\mathrm{sp}\, u$ is a closed subset of $\mathbb{T}$.

3.10 With a bit more, work, we can also show that for any $a \in A_{sa}$ we have $\mathrm{sp}(a) \subset \mathbb{R}$. First, note that for any element a in a unital Banach algebra $\sum_{n=0}^{\infty} \|a^n/n!\| \leq \sum_{n=0}^{\infty} \|a\|^n/n!$ and so converges. We set $e^a := \sum_{n=0}^{\infty} a^n/n!$. For any $a \in A$, one can check that the map

$$\phi : \mathbb{R} \to A : t \to e^{ta}$$

is differentiable at every $t \in \mathbb{R}$ with derivative $a\phi(t)$ and it satisfies $\phi(0) = e^0 = 1$. Moreover, these completely characterise $t \to e^{ta}$ in the sense that if ψ is another function with these properties then necessarily $\phi = \psi$. (Details: exercise; recall how this is done in the case that $A = \mathbb{R}$).

Using this characterisation, it follows that $e^{a+b} = e^a e^b$ for any $a, b \in A$ with $ab = ba$. In particular, e^a is always invertible.

3.11 Now let A be a unital C*-algebra and If a is self-adjoint, then e^{ia} is invertible of norm 1 hence it is a unitary. As we saw in 3.9, we must then have $\mathrm{sp}(e^{ia}) \subset \mathbb{T}$. Let $\lambda \in \mathrm{sp}(a)$. Let $b = \sum_{n=2}^{\infty} i^n (a - \lambda)^{n-1}/n!$. Note that b commutes with a. We have

$$e^{ia} - e^{i\lambda} = (e^{i(a-\lambda)} - 1)e^{i\lambda} = (a - \lambda)be^{i\lambda}.$$

Since b commutes with a and hence $(a - \lambda)$ and $(a - \lambda)$ is not invertible, we see that $e^{ia} - e^{i\lambda}$ is not invertible. Thus $e^{i\lambda} \in \mathrm{sp}(e^{ia}) \subset \mathbb{T}$ so we must have $\lambda \in \mathbb{R}$.

3.12 THEOREM: *Let A be a C*-algebra and let $a \in A_{sa}$. Then $r(a) = \|a\|$.*

PROOF: Exercise. ∎

Note that this means that the norm of an element in a in a C*-algebra A depends only on spectral data since: if $a \in A$ then a^*a is self-adjoint and

$$\|a\| = \|a^*a\|^{1/2} = (r(a))^{1/} = \left(\sup_{\lambda \in \mathrm{sp}(a)} |\lambda| \right)^{1/2}.$$

This gives us the next theorem.

3.13 THEOREM: *If the *-algebra A admits a norm which makes A into a C*-algebra, then it is the unique C-norm on A.*

Minimal unitisation. We saw in Exercise 1.14 that a nonunital Banach algebra A can be embedded into a unital Banach algebra $\tilde{A}$. There is also a way of defining a unitisation (in fact, more than one) of a nonunital C*-algebra. Unfortunately, we can't simply take $A \oplus \mathbb{C}$ with multiplication and norm as given in Exercise 1.14. The reason is that the norm there is not a C*-norm (check!). So we have to be a bit more careful in how we adjoin a unit to a nonunital C*-algebra.

3.14 If $T : A \to B$ is a linear operator, we equip it with the operator norm:

$$\|T\| := \sup_{a\in A, \|a\|\leq 1} \|T(a)\|_B,$$

which is just the usual norm for a linear operator if we consider A and B as Banach spaces.

3.15 A left multiplier L of A is a bounded linear operator $L : A \to A$ which satisfies $L(ab) = L(a)b$ for every $a, b \in A$. If $a \in A$, then a can act on A as a left multiplier $b \mapsto ab$. Similarly, it is easy to see that any $\lambda \in \mathbb{C}$ induces a left multiplier on A by $b \mapsto ab$.

3.16 Let $\tilde{A} = A \oplus \mathbb{C}$ as a vector space. We endow $\tilde{A}$ with the same multiplication as for the Banach algebra unitisation of A. Define an involution $^* : \tilde{A} \to \tilde{A}$ by $(a, \lambda)^* = (a^*, \overline{\lambda}))$. Now, to make A into a C*-algebra , we view the elements of $\tilde{A}$ as left operators on the A.

$$\|(a, \lambda)\|_{\tilde{A}} := \sup_{b\in A, \|b\|\leq 1} \{\|ab + \lambda b\|_A\}.$$

One then checks that this makes $\tilde{A}$ into a unital C*-algebra. Moreover, the map $a \mapsto (a, 0)$ identifies as an ideal in A. We call $\tilde{A}$ the *minimal unitisation of* A. What might be thought of as the "maximal" unitisation of A, the multiplier algebra, is discussed in Section 8. Note that this norm is the unique norm making $\tilde{A}$ into a C*-algebra and unless otherwise specified, this is the norm we use (rather than the one defined in Exercise 1.14) for $\tilde{A}$.

3.17 Example: Let $A = C_0(X)$ where X is a locally compact Hausdorff space. Then $\tilde{A} = C(X \cup \{\infty\})$ where $X \cup \{\infty\}$ is just the one point compactification of X []. In this way, we think of adjoining a unit as the noncommutative version of compactification.

The nice thing about the unitisation is that we will now be able to prove many theorems in the (usually easier) unital setting without any loss of generality.

3.18 Let A and B be C*-algebras. A **-homomorphism* $\phi : A \to B$ is an algebra homomorphism which is involution-preserving, $\phi(a^*) = \phi(a)^*$ for every $a \in A$. If A and B are unital, then, as before, we say that ϕ is a unital *-homomorphism if $\phi(1_A) = 1_B$.

If B is unital and $\phi : A \to B$ is a *-homomorphism then there exists a unique extension $\overline{\phi} : \tilde{A} \to B$ such that $\overline{\phi}$ is unital.

3.19 The following proposition is another nice implication of the C*-equality.

Proposition: *A *-homomorphism between C*-algebras is always norm-decreasing. In particular, it is always continuous.*

Proof: Let $\phi : A \to B$ be a *-homomorphism, and let $a \in A$. As we saw in 3.18, by replacing A and B by their unitisations if necessary, we can assume that ϕ, A and B are all unital. Since $\phi(1_A) = 1_B$, it is easy to check that if a is invertible so is $\phi(a)$ is invertible in B. It follows that $\mathrm{sp}(\phi(a)) \subset \mathrm{sp}(a)$. The result now follows from Theorem 3.12. ∎

3.20 Corollary: *Any *-isomorphism of* C*-*algebras is automatically isometric.*

3.21 Recall that $\Omega(A)$ is the spectrum, or character space, of A (see 2.10). If X is locally compact and Hausdorff and $x \in X$ then the $\mathrm{ev}_x(f) = f(x)$ is a character of $C_0(X)$. In fact, this is all of them.

Theorem: *Let $A = C(X)$ for some compact Hausdorff space X. Then*

$$\Omega : X \to \Omega(A) : x \to \mathrm{ev}_x$$

is a homeomorphism.

Proof: Let $(x_\lambda)_\Lambda$ is a net in X with $\lim_\lambda x_\lambda \to x \in X$. Then $\mathrm{ev}_{x_\lambda}(f) = f(x_\lambda) \to f(x) - \mathrm{ev}_x$ for every $f \in C(X)$, so $(ev_\lambda)_\Lambda$ is weak-* convergent to ev_x. Thus the map is continuous.

Suppose now that $x \neq y \in X$. Then by Urysohn's lemma, there is $f \in C(X)$ such that $f(x) = 1$ and $f(y) = 0$. Thus $\mathrm{ev}_x \neq \mathrm{ev}_y$ and so we see that the map is injective.

Now let us prove surjective:. Let τ be a character on $C(X)$. Let $M := \ker(\tau)$; this is a maximal, hence proper, ideal in $C(X)$. We show that M separates points. If $x \neq y$ the there is $f \in C(X)$ with $f(x) = 1$ and $f(y) = 0$. Now $f - \tau(f) \in M$ satisfies $f(x) - \tau(f) \neq f(y) - \tau(f)$ so by the Stone–Weierstrass theorem, there is $x \in X$ such that $f(x) = 0$ for every $f \in M$.

Thus $(f - \tau(f))(x) = 0$ and so $f(x) = \tau(f)$ for every $f \in C(X)$. It follows that $\mathrm{ev}_x = \tau$ and so Ω is surjective.

Since any continuous bijective map from a compact space is a homeomorphism, the result follows. ∎

3.22 Now we come to one of the most important results in C*-algebra theory, that for a commutative C*-algebra , the Gelfand Transform of 2.15 is an isometric *-isomorphism. This gives a complete characterisation of commutative C*-algebras : they are *always*, up to *-isomorphism, of the form $C_0(X))$ for some locally compact Hausdorff space X.

Theorem: [Gelfand–Naimark] *Let A be a commutative* C*-*algebra . Then the Gelfand transform*

$$\Gamma : A \to C_0(\Omega(A)) : a \to \hat{a}$$

*is an isometric *-isomorphism.*

Proof: If $\phi \in \Omega(A)$ then $\phi(a) \in \mathbb{R}$ whenever $a \in A_{sa}$. Thus for any $A \ni c = a+ib$ with $a, b \in A_{sa}$ we have $\phi((a+ib)^*) = \phi(a-ib) = \phi(a) - i\phi(b) = \overline{(\phi(a)+i\phi(b))}$, that is, ϕ is a *-homomorphism. It follows that $\hat{a^*}(\phi) = \overline{(\hat{a})(\phi)}$ for any $a \in A$ and any $\phi \in C_0(\Omega(A))$ meaning Γ is a *-homomorphism. This moreover implies that $\|\hat{a}\|^2 = \|\hat{a}^*\hat{a}\| = \|\widehat{a^*a}\| = r(a^*a) = \|a^*a\| = \|a\|^2$, so the map is isometric. Finally, to see that the map is also surjective, we appeal to the Stone–Weierstrass theorem: the image of Γ contains functions which don't simultaneously vanish any where on $\Omega(A)$ and also separates points, thus is exactly $C_0(\Omega(A))$. ∎

At first glance, Theorem 3.22 applies to the relatively small class of commutative C*-algebras. While it is true that in greater generality we don't have such an explicit characterisation for a class of C*-algebras, what we do get is an extremely useful tool: the continuous functional calculus for normal elements.

3.23 Theorem: *Let A be a unital C*-algebra and let $a \in A$ be a normal element. Then there is a map*

$$\gamma : C(\mathrm{sp}(a)) \to A : (z \mapsto z) \to a$$

is an isometric *-homomorphism and $\gamma(C(\mathrm{sp}(a))) = \mathrm{C}^*(a, 1)$

Proof: Since a is a normal element, $\mathrm{C}^*(a, 1)$ is abelian. Thus by Theorem 3.22 we have a *-isomorphism

$$\Gamma : \mathrm{C}^*(a, 1) \to C(\Omega(\mathrm{C}^*(a, 1))) : a \mapsto \hat{a}.$$

By Theorem 2.17, $h : \Omega(\mathrm{C}^*(a, 1)) \to \mathrm{sp}(a)$ is a homeomorphism so we also have an isomorphism $\psi : C(\mathrm{sp}(a)) \to C(\Omega(\mathrm{C}^*(a, 1)))$. Let $f(z) = z$ for $z \in \mathrm{sp}(a)$. Let $\gamma = \Gamma^{-1} \circ \psi$ Since $C(\mathrm{sp}(a))$ is generated by 1 and f, γ is the unique unital *-homorphism with $\gamma(f) = a$. Clearly γ is isometric and its image is $\mathrm{C}^*(a.1)$. ∎

3.24 Exercise: Write down and prove the nonunital version of Theorem 3.23.

The reason for highlighting the unital version is the following categorical theorem:

3.25 Theorem: *The correspondence between X and $C(X)$ is a categorical equivalence between the category of compact Hausdorff spaces and continuous maps to the category of unital C*-algebras and unital *-homomorphisms.*

3.26 This is where we get the nomer "noncommutative topology" for the study of C*-algebras. In general it useful to think of the C*-landscape as having two coasts at opposite ends, one of which consists of bounded operators on Hilbert spaces and matrix algebras, the other consisting of commutative C*-algebras . The interesting part of the theorem comes as we move inland, as most C*-algebras , of course, lie somewhere in between. Some of our best tools are brought in via either coast and it's often useful to keep these two examples in mind.

With this result in hand, we will be able to do a lot more in our C*-algebras . If $p \in \mathbb{C}[z_1, z_2]$ is a polynomial, then since A is an algebra, it is clear that

$p(a, a^*) \in C^*(a, 1)$. (In the nonunital case we would require that p have no constant terms; then $p(a, a^*) \in C^*(a, 1)$.) Since p of this form are dense in $C(\mathrm{sp}(a))$, we can then define Using Theorem 3.23 we define

$$f(a) := \gamma(f) \in C^*(a, 1),$$

or when A is nonunital, for $f \in C_0(\mathrm{sp}(a))$,

$$f(a) := \gamma(f) \in C^*(a).$$

The following is sometimes called the *spectral mapping theorem.*

3.27 THEOREM: *Let A be a C*-algebra and let $a \in A$ be a normal element. Then for any $f \in C_0(\mathrm{sp}(a))$, the element $f(a)$ is normal and we have*

$$\mathrm{sp}(f(a)) = f(\mathrm{sp}(a)).$$

Furthermore, if $g \in C_0(\mathrm{sp}(f(a))$ then $g \circ f(a) = g(f(a))$.

PROOF: Exercise. ∎

3.28 EXERCISE: Let A be a *-algebra. Two elements $a, b \in A$ are said to be *unitarily equivalent* if there exists a unitary $u \in A$ such that $uau^* = b$. Show that if a and b are unitarily equivalent then $\mathrm{sp}(a) = \mathrm{sp}(b)$.

EXERCISES

3.1 Let A be a *-algebra and let $a \in A$. Describe $\mathrm{sp}(a^*)$.

3.2 Let A be a C*-algebra and $p \in A$ a nonzero projection. What is $\mathrm{sp}(p)$?

3.3 Let $\psi : A \to \mathbb{R}$ be a differentiable map with derivative $a\psi(t)$ and $\psi(0) = 1$. Show that $\psi(t) = e^{ta}$.

3.4 A C*-algebra is *simple* if it has no nontrivial (closed, two-sided) ideals. Give an example of a finite-dimensional simple C*-algebra. Show that the compact operators $\mathcal{K}$ are a simple C*-algebra. (Here $\mathcal{K}$ denotes compact operators on a separable Hilbert space. Since there is only one separable Hilbert space up to isomorphism we often don't reference the underlying Hilbert space).

3.5 Let $A = C(X)$ where X is a compact Hausdorff space. Show that if $F \subset X$ is a closed subset then

$$\{f \in A \mid f|_F = 0\}$$

is an ideal. Show that every ideal in A has this form.

3.6 Describe all simple commutative C*-algebras.

3.7 Let $A \subset \mathcal{B}(H)$ be a concrete C*-algebra. Show that $M_n(A)$ admits a C*-norm and is a C*-algebra (ie. complete with respect to this norm).

3.8 Let A be a C*-algebra and let $a \in A_{sa}$. Then $r(a) = \|a\|$. Show that if A is a *-algebra admitting a C*-norm, then the norm is the unique C*-norm on A.

3.9 Let A be a unital C*-algebra. Suppose $a, b \in A$ are normal elements that are unitarily equivalent (that is, there exists a unitary $u \in \mathcal{U}(A)$ such that $u^*au = b$). Show that the C*-subalgebras $C^*(a, 1)$ and $C^*(b, 1)$ are isomorphic.

3.10 Let A be a unital algebra. Let $a \in A_{sa}$ and $0 < \epsilon < 1/4$. Suppose $\mathrm{sp}(a) \subset [0, \epsilon] \cup [1-\epsilon, 1]$. Show that there is a projection $p \in A$ with $\|p - a\| \leq \epsilon$.

3.11 Write down and prove the nonunital version of Theorem 3.23.

3.12 Let $\iota : (0,1) \hookrightarrow \mathbb{R}$ be the inclusion map. Then ι is continuous. Show that $\iota_* : C_0(\mathbb{R}) \to C_0((0,1)$ defined by $\iota_*(f) = f \circ \iota$ is not a *-homomorphism. What goes wrong?

3.13 Prove the spectral mapping theorem: Let A be a C*-algebra and let $a \in A$ be a normal element. Then for any $f \in C_0(\mathrm{sp}(a))$, the element $f(a)$ is normal and we have

$$\mathrm{sp}(f(a)) = f(\mathrm{sp}(a)).$$

Furthermore, if $g \in C_0(\mathrm{sp}(f(a))$ then $g \circ f(a) = g(f(a))$.

4. Positive elements

We'll give the functional calculus a workout in this section. The first thing is to define positive elements in a C*-algebra as well as a partial order on its self-adjoint elements.

4.1 Exercise: Let A be a *-algebra. Show that any element $a \in A$ can be written as $a = a_1 + ia_2$ where a_1, a_2 are self-adjoint. In this sense, the self-adjoint elements play the role of "real" elements in A, in analogy to real numbers in $\mathbb{C}$.

Since we're talking about self-adjoint elements, it might be useful to consider what is going on in the case of functions defined on an interval in $\mathbb{R}$.

4.2 Let A be a C*-algebra. An element $a \in A$ is said to be *positive* if $a \in A_{sa}$ and $\mathrm{sp}(a) \subset [0, \infty)$. The set of positive elements is denoted A_+.

If $a, b \in A_{sa}$ then we write $a \leq b$ if $b - a$ is positive. This defines a partial order on A_{sa}.

4.3 Lemma: *Let X be a locally compact Hausdorff space and let $f \in C_0(X)$ satisfy $f(x) \geq 0$ for all $x \in X$. Then f has a unique positive square root.*

4.4 The following fact often proves handy and makes use of the functional calculus.

Proposition: *Every positive element has a unique positive square root.*

Proof: Let $a \in A_+$. It is clear this holds if $a = 0$, so assume otherwise. Since $\mathrm{sp}(a) \subset [0, \infty)$ there is a function $f \in C(\mathrm{sp}(a))$, $f \geq 0$ which satisfies $(f(t))^2 = t$ for every $t \in \mathrm{sp}(a)$. Since a is normal, we can set $a^{1/2} := f(a)$, which is a positive square root for $a \in A$. Suppose there is another $b \in A_+$ with $b^2 = a$. Since $b \neq 0$

and b^2 commutes with a, so does b, and by approximating by polynomials, b also commutes with $a^{1/2}$. Thus the C*-algebra B generated by a and b is commutative, and moreover $\mathrm{sp}(a) \subset \Omega(B)$. Thus by uniqueness of the square-root in $C(\Omega)$, $b = a^{1/2}$. ∎

4.5 Proposition: *A unital* C*-*algebra is linearly spanned by its unitaries.*

Proof: We saw that every element can be written as a linear combination of self-adjoint elements, so we need only show that the self-adjoints are spanned by unitaries. Let $a \in A_{sa}$. By scaling if necessary, we may also assume that $\|a\| \leq 1$. In this case, $1 - a^2 \geq 0$ and so has a positive square root, $\sqrt{1-a^2}$. Let $u_1 = a - i\sqrt{1-a^2}$ and $u_2 = a + i\sqrt{1-a^2}$; it is easily checked that these are unitaires and that

$$a = u_1/2 + u_2/2.$$

∎

4.6 Recall that a convex cone is a subset of a vector space which is closed under linear combinations with positive coefficients.

Theorem: *The A_+ is closed convex cone.*

Proof: Without loss of generality, we may assume that A is unital. Let $a \in A_+$ and $\lambda \geq 0$. Then clearly $\lambda a \geq 0$ so we only need to check that the sum of two positive elements is again positive. Let $a, b \in A_+$. We claim that if $f \in C(X)$ where X is a compact subset of $\mathbb{R}$ that f is positive if there is $r \in \mathbb{R}_{\geq 0}$ such that $\|f - r\| \leq r$. If $r = 0$ we must have $f = 0$, which is positive. Suppose that $r > 0$ but f is not positive. Then there is some $t \in X$ such that $f(t) < 0$. Then $|f(t) - r| > r$ so $\|f - r\| = \sup_{t \in X} |f(t) - r| > r$, a contradiction. This proves the claim.

Note that furthermore, if f is positive then $\|f - r\| < r$ if $r \leq \|f\|$

Since $a + b$ is self-adjoint, so we may identify $\mathrm{C}^*(a + b, 1) \cong C(\mathrm{sp}(a + b))$ and $a + b$ with $f(t) = t$. Thus we need only find such an r satisfying $\|a + b - r\| \leq r$. Let $r = \|a\| + \|b\|$. We have that $\|a - \|a\|\| \leq \|a\|$ and $\|b - \|b\|\| \leq \|b\|$ by the above. Thus

$$\|a + b - \|a\| - \|b\|\| = \|a - \|a\| + b - \|b\|\| \leq \|a - \|a\|\| + \|b - \|b\| \leq \|a\| + \|b\|.$$

Hence $a + b$ is positive.

To show that it is closed, first note that by the above

$$B := \{a \in A_+ \mid \|a\| \leq 1\} = \{a \mid \|a - 1\| \leq 1\} \cap A_{sa}.$$

Both $\{a \mid \|a - 1\| \leq 1\}$ and A_{sa} are closed, thus $\mathbb{R}B = \mathbb{R}A_+ = A_+$ is closed. ∎

4.7 Let $a \in A_{sa}$. Let $a_- \in C(\mathrm{sp}(a))$ be the function $a(t) = -t$ when $t \leq 0$ and 0 otherwise. Denote by $a_+ \in C(\mathrm{sp}(a))$ the function $a_+(t) = t$ for every $t \geq 0$ and 0 otherwise. Then $a_+, a_- \in A_+$ and $a = a_+ - a_-$. Note also that $a_+ a_- = 0$.

The above observation also means that $a_+ - a = a_- \in A_+$ and $a \leq a_+$. This, together with the previous theorem, gives us the next corollary.

COROLLARY: *$(A_{sa}, \leq)$ is upwards directed.*

4.8 THEOREM: *For every $a \in A$, the element a^*a is positive.*

PROOF: Clearly $a^*a \in A_{sa}$. Suppose that $-a^*a \in A_+$. Then by 1.4, so is $-aa^*$. We have $a = b + ic$ for some $b, c \in A_{sa}$, so $a^*a = b^2 + ibc - icb + c^2$ and $aa^* = b^2 - ibc + icb + c^2$. Hence $a^*a = 2b^2 + 2c^2 - aa^*$ is the sum positive elements and must also be positive. Thus $\|a^*a\| = 0$, hence $\|a\|^2 = 0$ and therefore $a = 0$.

Suppose that $a \neq 0$. Then by the above, $-a^*a$ is not positive. As we saw in 4.7, we can write $a^*a = b - c$ where $b, c \in A_+$ and $bc = 0$. We will use the above to show that $c = 0$. Consider ac. We have $-(ac)^*(ac) = -ca^*ac = -c(b - c)c = c^3 \in A_+$, hence $ac = 0$ and finally $c = 0$. ∎

The next theorem gives some more useful facts about the self-adjoint and the positive elements.

4.9 THEOREM: *Let A be a C*-algebra . Let $a, b \in A_{sa}$ with $a \leq b$. Then*

(i) *for any $c \in A$, we have $c^*ac \leq c^*bc$;*
(ii) *$\|a\| \leq \|b\|$;*
(iii) *if A is unital and $a, b \in \text{Inv}(A)$ then $b^{-1} \leq a^{-1}$.*

PROOF:

(i) Since $a \leq b$, the element $b - a \in A_+$ and therefore has a positive square rooot. Then $c^*bc - c^*ac = c^*(b - a)c = ((a - b)^{1/2}c)^*((a - b)^{1/2}c)$, which is positive by Proposition 4.8.
(ii) Without loss of generality, we may assume that A is unital. Then, using the functional calculus, we have $b \leq \|b\|$, so also $a \leq \|b\|$. Then, since $C^*(a, 1) \to C(\text{sp}(a))$ is isometric, we have that $\|a\| \leq \|b\|$.
(iii) For any $a \in A$, if $c \in \text{Inv}(A)$ then $\lambda \in \text{sp}(c)$ if and only if $\lambda^{-1} \in \text{sp}(c)$ so $a^{-1}, b^{-1} \in A_{sa}$. By (i), $b^{-1/2}ab^{-1/2} \leq b^{-1/2}bb^{-1/2} = 1$. So, using (ii), $\|b^{-1/2}a^{1/2}\|^2 < 1$. Thus $\|a^{1/2}b^{-1}a^{1/2}\| \leq 1$ and $a^{1/2}b^{-1}a^{1/2} \leq 1$. By (i) again, mulitiplying $a^{-1/2}$ on either side gives $b^{-1} < a^{-1}$.

∎

We also have the next theorem, which we include separately because the proof is a little trickier.

4.10 THEOREM: *Let A be a C*-algebra and let $a, b \in A_+$ with $a \leq b$. Then $a^n \leq b^n$ for all $0 < n \leq 1$.*

PROOF: Let $a, b \in A_+$ with $a \leq b$. Let $\epsilon > 0$ and put $c = b + 1_{\tilde{A}}\epsilon$. Then $a \leq c$ and c is invertible in $\tilde{A}$. Set

$$S = \{n \in (0, \infty) \mid a^n \leq c^n\}.$$

Note that $1 \in S$. Suppose that $(t_n)_{n\in\mathbb{N}}$ is a sequence in S converging to $t \in (0, \infty)$. Then $\lim_{n\to\infty} a^{t_n} - c^{t_n} = a^t - c^t \geq 0$ since A_+ is closed.

Since $c \geq 0$, so is c^{-1} and thus has a positive square root. By Theorem 4.9 (i), this implies that $c^{-1/2}ac^{-1/2} \leq 1_{\tilde{A}}$, and, using the C*-equality that $\|a^{1/2}c^{-1/2}\| \leq 1$. We saw earlier that $\mathrm{sp}(xy) \setminus \{0\} = \mathrm{sp}(yx) \setminus \{0\}$ for any $x, y \in A$. Thus

$$\|c^{-1/4}a^{1/2}c^{-1/4}\| = r(c^{-1/4}a^{1/2}c^{-1/4}) = r(a^{1/2}c^{-1/2}) \leq 1,$$

so $c^{-1/4}a^{1/2}c^{-1/4} \leq 1$ and $a^{1/2} \leq c^{1/2}$. Thus $1/2 \in S$.

One shows similarly that if $s, t \in S$ then $(s + t)/2 = z \in S$. Indeed,

$$\begin{aligned} \|c^{-z/2}a^z c^{-z/2}\| &= r(c^{-z/2}a^z c^{z/2}) \\ &= r(c^{-s/2}a^{s/2}a^{t/2}c^{-t/2}) \\ &= \|(c^{-s/2}a^{s/2})(a^{t/2}c^{-t/2})\| \\ &\leq \|c^{-s/2}a^{s/2}\|\|a^{t/2}c^{-t/2}\| \\ &\leq 1. \end{aligned}$$

Thus $(0, 1] \subset S$. Since ϵ was arbitrary, we have that $a^n \leq b^n$ for every $n \in (0, 1]$. ∎

4.11 What happens for $n \geq 1$? If $A = C(X)$ is commutative and $f, g \geq 0$ with $f \leq g$, then certainly $f^2 \leq g^2$. In this case, however, an attempt to look to the commutative algebras leads us astray. Let us consider things from the other side, then, in spirit of 3.26. In fact, we need only consider M_2 to see that things can go wrong. The conclusion of the last theorem no longer necessarily hold. Let

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

Then $a \leq a + b$ but

$$(a + b)^2 - a^2 = ab + ba + b^2 = a + 2ab + b = \begin{pmatrix} 3/2 & 1 \\ 1 & 1/2 \end{pmatrix}$$

is not positive.

By the way, this above example is also a good illustration of the following: It is often very useful to look to the 2×2 matrices as a test case when deciding whether or not something holds in a C*-algebra.

Approximate units and hereditary C*-subalgebras. We have already seen that it is always possible to adjoin a unit to a nonunital C*-algebra. In many cases, however, this is not necessarily useful. For example, if one is investigating simple C*-algebras (no nontrivial ideas), then attaching a unit destroys simplicity. We can, however, often find an *approximate* unit. Let's consider continuous functions on some open interval I in $\mathbb{R}$, for example. Then the C*-algebra $C_0(I)$ is of course nonunital. However, for any finite subset of functions $F \subset C_0(I)$ nad any $\epsilon > 0$, we can always find another function, g such that $\|gf - f\| <$ for every $f \in F$. Since $C_0(I)$ is separable, one can find a nested increasing sequence of these finite subsets $F_0 \subset F_1 \subset \cdots$ that exhausts the algebra. Then, for every $n \in \mathbb{N}$ find g_n such that $fg_n = f$ for every $f \in F_n$. This gives us a sequence $(g_n)_{N\in\mathbb{N}} \subset C_0(I)$ which itself will not converge, but which still satisfies

$$\lim_{n\to\infty} g_n f = f \text{ for every } f \in C_0(I).$$

In the completely noncommutative case, consider the compact operators $\mathcal{K}(H)$ on a separable Hilbert space H with orthonormal $(e_i)_{i\in\mathbb{N}}$. They are of course nonunital but, thinking of $\mathcal{K}$ as infinite matrices, then if p_n is the projection $\sum_{i=1}^n e_i$ we also get a sequence that won't converge, but, for any $a \in \mathcal{K}$ we have

$$\lim_{n\to\infty} p_n a = \lim_{n\to\infty} a p_n = a.$$

4.12 Definition: Let A be a C*-algebra . An *approximate unit* for A is an increasing net $(u_\lambda)_{\lambda\in\Lambda}$ of positive elements such that

$$\lim_\lambda u_\lambda a = \lim_\lambda a u_\lambda = a,$$

for every $a \in A$.

4.13 Lemma: *Let A be a C*-algebra and let $A^1_+ := \{a \in A \mid a \in A_+, \|a\| < 1\}$. Then the set of $(A^1_+, \leq)$ is upwards-directed.*

Proof: First, let $a, b \in A_+$. Then $(1 + a), (1 + b) \in \mathrm{Inv}(\tilde{A})$. Suppose that $a \leq b$. Then $1 + a \leq 1 + b$ so $(1 + b)^{-1} \leq (1 + a)^{-1}$ and $a(1 + a)^{-1} = 1 - (1 + a)^{-1} \leq 1 - (1 + b)^{-1} = b(1 + b)^{-1}$. Note that $a(1 + a)^{-1}$ and $b(1 + b)^{-1}$ are both in A^1_+.

Now suppose that $a, b \in A^1_+$. Let $x = a(1 - a)^{-1}$ and $y = b(1 - b)^{-1}$ and set $c = (x + y)(1 + x + y)^{-1}$. Since $x \leq x + y$ we have that $a = x(x + 1)^{-1} \leq c$. Similarly $b \leq c$. ∎

4.14 Theorem: *Every C*-algebra A has an approximate unit. If A is separable, then A has a countable approximate unit.*

Proof: Let Λ be the upwards directed set $(A^1_+, \leq)$ and put $u_\lambda = \lambda$ for each $\lambda \in \Lambda$. Then $(u_\lambda)_\Lambda$ is an increasing net of positive elements with norm less than 1. We must show that $\lim_\lambda u_\lambda a = \lim_\lambda a u_\lambda = a$ for every $a \in A$. It is enough to show that this holds when $a \in A_+$. Let $\epsilon > 0$. Let $\Gamma : \mathrm{C}^*(a) \to C_0(X))$ be the Gelfand transform. Let $f = \Gamma(a)$ and let $K = \{x \in X | f(x)| \geq \epsilon\}$.

Let $\delta > 0$ such that $\delta < 1$ and $1 - \delta < \epsilon$. Let $g_\delta(x) = \delta$ if $x \in K$ and vanishing outside K (this is possible since K is compact). Then $g_\delta \in C_0(X)^1_+$ and $\|g_\delta f - f\| < \epsilon$. Since Γ is isometric, $\Gamma^{-1}(g_\delta) = \mu$ for some $\mu \in \Lambda$ and $\|u_\mu a - a\| = \|a u_\mu - a\| < \epsilon$.

Suppose that $\lambda \in \Lambda$ satisfies $\mu \leq \lambda$. Then $1 - u_\lambda \geq 1 - u_\mu$ so $a^{1/2}(1 - u_\lambda)a^{1/2} \leq a^{1/2}(1 - u_\mu)a^{1/2}$ and hence $\|a^{1/2}(1 - u_\lambda)a^{1/2}\| \leq \|a^{1/2}(1 - u_\mu)a^{1/2}\|$ and applying the C*-equality, $\|a - a u_\lambda\| < \|a - a u_\mu\| < \epsilon$. Hence $\lim_\lambda a u_\lambda = \lim_\lambda u_\lambda = a$.

The proof that if A is separable then A admits a countable approximate unit is an exercise. ∎

The following is a very useful theorem. We omit the proof for now because it uses some techniques that have not yet been described, but will return to it in the next section.

4.15 Theorem: *Let A be a* C*-*algebra and I a closed ideal in A. Then I has a* quasicentral approximate unit *for A, that is, I has an approximate unit $(u_\lambda)_{\lambda \in \Lambda}$ satisfying*

$$\|u_\lambda a - a u_\lambda\| \to 0 \text{ as } \lambda \to \infty$$

for every $a \in A$.

4.16 We have already defined C*-subalgebras as well as ideals. There is another important substructure in C*-algebras coming from the order structure.

Definition: A C*-subalgebra $B \subset A$ is called *hereditary* if, whenever $b \in B$ and $a \leq b$, then $a \in B$.

4.17 This next example is called a *corner* of a C*-algebra .

Theorem: *Let $p \in A$ be a projection. Then $pAp = \{pap \mid a \in A\}$ is a hereditary subalgebra of A.*

Proof: Since p is a projection, it is easy to check that pAp is C*-subalgebra. We may assume that $a \in pAp$ is positive since if $b \leq a$ then $b \leq a_+$. Suppose first that $b \leq a$ with b also positive or $b = -c$ where c is positive. Then $(1-p)b(1-p) \leq (1-p)a(1-p) = 0$. So $\|(1-p)b(1-p)\| = \|(1-p)b^{1/2}\|^2$ and we see that $pb = pb^{1/2}b^{1/2} = b^{1/2}b^{1/2} = b$. Similarly, $bp = b$, hence $b \in pAp$. If $b = b_+ - b_-$ with b_+, b_- positive and $b_+ b_- = b_- b_+ = 0$. Since $b \leq a$ we have that $b_+^2 \leq b_+^{1/2} a b_+^{1/2}$ and $b_+^2/\|b\| \leq (b_+^{1/2}/\|\|b^{1/2}\|)a(b_+^{1/2}/\|b^{1/2}\|) \leq a$. Since $b_+^2/\|b\|$ is positive, $b_+^2/\|b\| \in pAp$. Hence $b_+ \in pAp$. Then since $-b_- \leq b \leq a \in pAp$, we also have $b_- \in pAp$ and so $b \in pAp$, as required. ∎

For a positive element, we have the following generalisation:

4.18 Theorem: *Let $a \in A_+$. Then $a \in \overline{aAa}$ and $\overline{aAa}$ is the hereditary* C*-subalgebra generated by a. If B is a separable hereditary* C**-algebra, then $B = \overline{aAa}$ for some $a \in A_+$.*

Proof: It is clear that $\overline{aAa}$ is a C*-subalgebra. Let $(u_\lambda)_\Lambda$ be an approximate unit for A. Then $\lim_\lambda au_\lambda a = a^2$ so $a^2 \in \overline{aAa}$. Thus $C^*(a^2) \subset \overline{aAa}$ and by uniqueness of the positive square root, also $a \in \overline{aAa}$.

The proof that $\overline{aAa}$ is hereditary is similar to the case for a corner, so is left as an exercise.

Now suppose that $B \subset A$ is hereditary and separable. Since B is separable, it contains a countable approximate unit, say $(u_n)_{n\in\mathbb{N}}$. Let $a = \sum_{n=1}^\infty 2^{-n}u_n$. Then $a \in B$ and $a \geq 0$. Thus $\overline{aAa} \subset B$. For each $\mathbb{N} \setminus \{0\}$ we have that $2^{-n}u_n \leq a$ so $u_n \in \overline{aAa}$. Thus, if $b \in B$ we have $b = \lim_{n\to\infty} u_n b u_n$ where each $u_n b u_n \in \overline{aAa}$ hence $b \in \overline{aAa}$ and so we have shown that $B = \overline{aAa}$. ▮

4.19 Lemma: *Let L be a closed left ideal of A. Then L has a left approximate unit, that is, $(u_\lambda)_{\lambda\in\Lambda} \subset I$ with*

$$\lim_\lambda au_\lambda = a$$

for every $a \in A$.

Proof: Observe that $L \cap L^*$ is a C*-subalgebra of A and thus by Theorem 4.14 has an approximate unit $(u_\lambda)_{\lambda\in\Lambda}$. Let $a \in L$. Then $a^*a \in L \cap L^*$. Since

$$\|a - au_\lambda\|^2 = \|a^*a - a^*au_\lambda - u_\lambda a^*a + u_\lambda a^*au_\lambda\|,$$

we have

$$\|a - \lim_\lambda au_\lambda\|^2 = 0.$$

Thus $a = \lim_\lambda au_\lambda$. ▮

4.20 Theorem: *Let A be a* C**-algebra . There is a one-to-one correspondence between closed left ideals of A and hereditary subalgebras of A given by*

$$I \mapsto I^* \cap I; \qquad B \mapsto \{a \in A \mid a^*a \in B\}.$$

Proof: Let I be a closed left ideal in A and suppose $b \in (I \cap I^*)_+$. Since I is a closed left ideal, it contains a left approximate unit (u_λ). If $a \in A_+$ satisifies $a \leq b$, then $(1-u_\lambda)a(1-u_\lambda) \leq (1-u_\lambda)b(1-u_\lambda)$. Then

$$\begin{aligned} \|a^{1/2}(1-u_\lambda)\|^2 &= \|(1-u_\lambda)a(1-u_\lambda)\| \\ &\leq \|(1-u_\lambda)b(1-u_\lambda)\| \\ &= \|b^{1/2} - b^{1/2}u_\lambda\|^2 \\ &\to 0. \end{aligned}$$

Thus $\|a^{1/2}(1-u_\lambda)\|^2 \to 0$ and so $a^{1/2}$ is a limit of elements in I. It follows that $a^{1/2}$, and hence $a \in I$. Thus $I^* \cap I$ is a hereditary C*-subalgebra of A.

Now suppose that B is a hereditary C*-subalgebra. Let $I = \{a \in A \mid a^*a \in B\}$. Let $a \in I$, $b \in A$ and without loss of generality, assume $\|b\| \leq 1$. Then $(ab)^*(ab) = b^*a^*ab \leq a^*a$, so $ab \in I$ since B is hereditary. It is clear that I is closed since B is; thus I is a closed left ideal.

Finally, it is easy to check that the maps are mutual inverses. ∎

4.21 Corollary: *Every closed ideal is hereditary.*

Proof: Let $I \subset A$ be a closed ideal. Then in particular I is a closed left ideal and we have $I = I \cap I^*$ since if $a \in I$ and $(u_\lambda)_\Lambda$ is a left approximate unit for I then $a^* = \lim_\lambda (au_\lambda)^* = \lim_\lambda u_\lambda a^* \in I$. ∎

A hereditary C*-subalgebra often inherits properties of the C*-algebra itself. We shall see more examples later, but for now we can show the following

4.22 Theorem: *If A is a simple* C*-*algebra, then so is every hereditary subalgebra $B \subset A$.*

Proof: We claim that every ideal $J \subset B$ is of the form $I \cap B$ for some ideal $I \subset A$. Suppose that I is an ideal in A and let $b \in I \cap B$. Clearly $I \cap B$ is closed under addition and scalar multiplication and is norm-closed. Let $c \in B$. Without loss of generality assume that $\|c\| \leq 1$ and that $c \geq 0$. In that case $cb \leq b$ and similarly $bc \leq b$. Thus $cb, bc \in I \cap B$.

Now suppose that $J \subset B$ is an ideal. Let $(u_\lambda)_\Lambda$ be a quasicentral approximate unit for J. Set $I = \{au_\lambda \mid a \in A, \lambda \in \Lambda\}$. Then $J = I \cap B$; the details are easy to check. ∎

Exercises

Let A be a C*-algebra

4.1 A *partial isometry* in a C*-algebra is an element satisfying $v = vv^*v$ and $v^* = v^*vv^*$.

(a) Show that v^*v and vv^* are projections.

(b) Let $A = M_n$ and let p, q be projections. If $\mathrm{tr}(p) \leq \mathrm{tr}(q)$ show that there is a partial isometry $v \in M_n$ such that $v^*v = p$ and $vv^* \leq q$. (Here tr denotes the normalised trace on M_n, i.e. $\mathrm{tr}((a_{ij})_{ij}) = \frac{1}{n}\sum_{i=1}^n a_{ii}$.)

(c) Projections p and q are *Murray–von Neumann equivalent* if there is a partial isometry $v \in A$ with $v^*v = p$ and $vv^* = q$. Check that this is an equivalence relation. If $A = M_n$ describe the equivalence classes.

4.2 Let A be a separable C*-algebra. For two positive elements a, b in A we say that a is *Cuntz subequivalent* b and write $a \precsim b$ if there are $(v_n)_{n\in\mathbb{N}} \subset A$ such that $\lim_{n\to\infty} \|v_n b v_n^* - a\| = 0$. We write $a \sim b$ and say a and b are *Cuntz equivalent* if $a \precsim b$ and $b \precsim a$.

(a) Show that $\sim$ is an equivalence relation on the positive elements.

(b) Show that if p and q are projections then they are Cuntz equivalent if they are Murray–von Neumann equivalent.

(c) Let $f, g \in C(X)_+$ where X is a locally compact metric space. Show that if $\mathrm{supp}(f) \subset \mathrm{supp}(g)$ then for any $\epsilon > 0$ there is a positive function $e \in C(X)$ such that $\|f - ege\| < \epsilon$. Now show that $f \precsim g$ if and only if $\mathrm{supp}(f) \subset \mathrm{supp}(g)$.

(d) Let $a \in A_+$ where a is a separable C*-algebra. Show that $a \precsim a^n$ for every $n \in \mathbb{N}$. Use this to show that $a^*a \sim aa^*$ for every $a \in A$.

4.3 Let A and B be concrete C*-algebras. A linear map $\phi : A \to B$ is *positive* if $\phi(A_+) \subset B_+$. For any map $\phi : A \to B$ and $n \in \mathbb{N}$ we can define $\phi^{(n)} : M_n(A) \to M_n(B)$ by applying ϕ entry-wise. If $\phi^{(n)} : M_n(A) \to M_n(B)$ is positive for every $n \in \mathbb{N}$ then we say ϕ is *completely positive*.

(a) Show that a *-homomorphism $\phi : A \to B$ is completely positive.

(b) Let $A = B = M_2$ and let $\tau : M_2 \to M_2$ be the map taking a matrix to its transpose. Show that τ is positive but not completely positive. (Note that we need not restrict ourselves to concrete C*-algebras, but we haven't yet shown that $M_n(A)$ is actually a C*-algebra when we only know A is abstract.)

(c) Let $\phi : A \to B$ be a *-homomorphism and let $v \in B$. Show that the map

$$v^*\phi v : A \to B : a \mapsto v^*\phi(a)v$$

is completely positive.

4.4 Let X be a compact metric space. Show that for any finite subset $\mathcal{F} \subset C(X)$ and any $\epsilon > 0$ there is a finite dimensional C*-algebra F and completely positive contractive (c.p.c.) maps $\psi : C(X) \to F$ and $\phi : F \to C(X)$ such that $\|\phi \circ \psi(f) - f\| < \epsilon$ for every $f \in \mathcal{F}$.

4.5 Suppose $d \in \mathbb{N}$ and that X has covering dimension d, that is, every open cover $\mathcal{U}$ of X has a finite subcover $\mathcal{U}'$ such that $\sum_{U\in\mathcal{U}'} \chi_U(x) \leq d+1$ for every $x \in X$. (Here χ_U is the indicator function on U.) Show that for every finite subset $\mathcal{F} \subset C(X)$ and every $\epsilon > 0$ there are finite dimensional C*-algebras $F_0, \dots F_d$ and c.p.c. maps $\psi : C(X) \to \bigoplus_{i=0}^d F_i$ and $\phi_i : F_i \to C(X)$ such that $\|(\oplus_{i=0}^d \phi_i) \circ \psi(f) - f\| < \epsilon$ for every $f \in \mathcal{F}$ and moreover that $\phi_i(f)\phi_i(g) = \{0\}$ whenever $f \neq g$.

4.6 Prove that every separable C*-algebra admits a countable approximate unit.

4.7 Show that if $a \in A_+$ then $\overline{aAa}$ is a hereditary C*-subalgebra.

4.8 Let A be a unital C*-algebra, $a \in \mathrm{Inv}(A)$ and $p \in A$ a projection. Show that if a commutes with p then a is invertible in the corner pAp. If $a \in A$ is invertible in pAp, is $a \in \mathrm{Inv}(A)$?

5. Positive linear functionals and representations

5.1 A linear map $\phi : A \to B$ between C*-algebras is called *positive* if $\phi(A_+) \subset B_+$. Note that a *-homomorphism is always positive. If $\phi : A \to \mathbb{C}$ is positive, it is called a *positive linear functional.* Any positive linear functional ϕ also satisifies $\phi(A_{sa}) \subset \mathbb{R}$ and hence $\phi(a^*) = \overline{\phi(a)}$ for every $a \in A$.

If ϕ is a positive linear functional with $\|\phi\| = 1$, then ϕ is called a *state* and if it satisfies $\phi(ab) = \phi(ba)$ for every $a, b \in A$ then it is called a *tracial state.* The set of states, respectively tracial states, on A will be denoted by $S(A)$, respectively $T(A)$.

5.2 Example: Let $A = M_n$. Then the usual normalised trace, $\mathrm{tr} : A \to \mathbb{C}$ given by

$$\mathrm{tr}((a_{ij})_{ij}) = \frac{1}{n} \sum_{i=1}^{n} a_{ii}$$

is a tracial state.

5.3 Example: Let $A = C(X)$ be a commutative C*-algebra. Then any character is a tracial state, but not every tracial state is of this form. Suppose that μ is a probability measure on X. Then the map $\tau : A \to \mathbb{C}$ given by

$$\tau(f) = \int f d\mu$$

is a tracial state. One often thinks of a tracial state as a noncommutative measure.

5.4 Example: Let H be a Hilbert space and $\xi \in H$ a nonzero vector. Then

$$\phi(a) = \langle a\xi, \xi \rangle$$

is a postive linear functional, but not necessarily tracial.

The next theorem is another example of how C*-algebras are in general more well-behaved than arbitrary Banach algebras.

5.5 Theorem: *Any positive linear functional on a* C**-algebra is bounded.*

Proof: Suppose not. Let $\phi : A \to \mathbb{C}$ be an unbounded linear functional on some C*-algebra A. Since ϕ is unbounded, we can find a sequence $(a_n)_{n \in \mathbb{N}}$ of elements in the unit ball of A such that $|\phi(a_n)| \to \infty$ as $n \to \infty$. Without loss of generality, we may assume that each $a_n \in A_+$, for if ϕ was bounded on every $a_n \in A_+$ then ϕ would be bounded everywhere. Passing to a subsequence if necessary, we may further assume that for every $n \in \mathbb{N}$ we have $\phi(a_n) \geq 2^n$. Let $a = \sum_{n\in\mathbb{N}} 2^{-n} a_n \in A_+$. Then, for every $N \in \mathbb{N}$,

$$\phi(a) = \sum_{n\in\mathbb{N}} 2^{-n}\phi(a_n) \geq \sum_{n\in\mathbb{N}} 1 > N,$$

which is impossible. ∎

Positive linear functionals admit the following Cauchy–Schwarz inequality.

5.6 Proposition: *Let $\phi : A \to \mathbb{C}$ be a positive linear functional on the* C*-*algebra A. Then*

$$|\phi(a^*b)|^2 \leq \phi(a^*a)\phi(b^*b)$$

for every $a, b \in A$.

Proof: We may obviously assume that $\phi(a^*b) \neq 0$. Since ϕ is positive, for any $\lambda \in \mathbb{C}$ we have $\phi((\lambda a + b)^*(\lambda a + b)) \geq 0$. In particular, this holds for

$$\lambda = t\frac{|\phi(a^*b)|}{\phi(b^*a)},$$

for any $t \in \mathbb{R}$, giving

$$t^2\phi(a^*a) + 2t|\phi(a^*b)| + \phi(b^*b) \geq 0.$$

If we have $\phi(a^*a) = 0$ then $\phi(b^*b) \geq 2t|\phi(a^*b)|$ for every $t \in \mathbb{R}$, which is impossible unless $|\phi(a^*b)|$ is also zero; in this case the inequality holds. If $\phi(a^*a) \neq 0$ then let

$$t = -\frac{|\phi(a^*b)|}{\phi(a^*a)}.$$

Then

$$\frac{|\phi(a^*b)|^2}{\phi(a^*a)} - 2\frac{|\phi(a^*b)|^2}{\phi(a^*a)} + \phi(b^*b) \geq 0,$$

from which the result follows. ∎

5.7 Proposition: *Let A be a* C*-*algebra and let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit. Let $\phi \in A^*$. Then ϕ is positive if and only if* $\lim_\lambda \phi(u_\lambda) = \|\phi\|$. *In particular if A is unital then $\phi(1) = \|\phi\|$.*

Proof: Suppose that ϕ is positive. Then, since $(u_\lambda)_\Lambda$ is an increasing net of positive elements, $(\phi(u_\lambda))_\Lambda$ is increasing in $\mathbb{R}_+$. Since it's moreover bounded, it converges to some $r \in \mathbb{R}_+$. Since each u_λ has norm less than one, $r \leq \|\phi\|$. Now if $a \geq 0$ with $\|a\| \leq 1$, then, using the Cauchy–Schwarz inequality,

$$|\phi(au_\lambda)|^2 \leq \phi(a^*a)\phi(u_\lambda^2) \leq \phi(a^*a)\phi(u_\lambda) \leq r\phi(a^*a).$$

Since ϕ is continuous and $au_\lambda \to a$, it follows that

$$|\phi(a)|^2 \leq r\phi(a^*a) \leq r\|\phi\|,$$

hence $\|\phi\|^2 \leq r\|\phi\|$ which is to say $\lim_\lambda \phi(u_\lambda) = \|\phi\|$.

For the converse, first let $a \in A_{sa}$. We will show that $\phi(a) \in \mathbb{R}$. Let α and β be real numbers such that $\phi(a) = \alpha + i\beta$. Without loss of generality, assume that $\beta \geq 0$. For $n \in \mathbb{N}$ let λ be sufficiently large that $\|u_\lambda a - a u_\lambda\| < 1/n$. Then

$$\begin{aligned} \|nu_\lambda - ia\|^2 &= \|nu_\lambda^2 - a^2\| \\ &= \|n^2 u_\lambda^2 + a^2 - in(u_\lambda a - a u_\lambda)\| \\ &\leq n^2 + 2. \end{aligned}$$

We also have that

$$\begin{aligned} \lim_\lambda |\phi(nu_\lambda - ia)|^2 &= (n\|\phi\| + \beta - i\alpha)(n\|\phi\| + \beta + i\alpha) \\ &= (n\|\phi\| + \beta)^2 + \alpha^2. \end{aligned}$$

Thus

$$\begin{aligned} (n\|\phi\| + \beta)^2 + \alpha^2 &= \lim_\lambda |\phi(nu_\lambda - ia)|^2 \leq \|\phi\|^2 \|nu_\lambda - ia\|^2 \\ &\leq \|\phi\|^2 (n^2 + 2), \end{aligned}$$

and then

$$n^2\|\phi\|^2 + 2n\|\phi\|\beta + \beta^2 + \alpha^2 \leq n^2\|\phi\|^2 + 2,$$

for every $n \in \mathbb{N}$. Since $\beta^2 + \alpha^2 \geq 0$ and we assumed that $\beta \geq 0$, for large enough n this can only hold with $\beta = 0$. Thus $\phi(a) \in A_{sa}$.

Now if $a \in A_+$ and $\|a\| \leq 1$, then $u_\lambda - a \in A_{sa}$ and $u_\lambda - a \leq u_\lambda$ so $\lim_\lambda \phi(u_\lambda - a) \leq \|\phi\|$. Thus $\phi(a) \geq 0$. ∎

5.8 COROLLARY: *Let A be a nonunital C*-algebra. Then any positive linear functional $\phi : A \to \mathbb{C}$ admits a unique extension $\tilde{\phi} : \tilde{A} \to \mathbb{C}$ with $\|\tilde{\phi}\| = \|\phi\|$.*

PROOF: Exercise. ∎

5.9 PROPOSITION: *Let A be a C*-algebra and let $B \subset A$ a C*-subalgebra. Every positive linear functional $\phi : B \to \mathbb{C}$ admits an extension $\tilde{\phi} : A \to \mathbb{C}$. If $B \subset A$ is a hereditary C*-subalgebra then the extension is unique and if $(u_\lambda)_\Lambda$ is an approximate unit for B then*

$$\tilde{\phi}(a) = \lim_\lambda \phi(u_\lambda a u_\lambda),$$

for any $a \in A$.

PROOF: By the previous corollary, we may suppose that A is unital and $1_A \in B$. Then, applying the Hahn-Banach theorem there is an extension $\tilde{\phi} \in A^*$ such that $\|\tilde{\phi}\| = \|\phi\|$. Since $\|\phi\| = \phi(1) = \tilde{\phi}(1)$, it follows from Proposition 5.7 that $\tilde{\phi}$ is also positive.

If B is hereditary, then $\lim_\lambda \phi(u_\lambda) = \|\phi\| = \|\tilde{\phi}\| = \tilde{\phi}(1)$. Thus $\lim_\lambda \tilde{\phi}(u_\lambda - 1) = 0$. For any $a \in A$,

$$\begin{aligned} |\tilde{\phi}(a) - \phi(u_\lambda a u_\lambda)| &\leq |\tilde{\phi}(a - a u_\lambda)| + |\tilde{\phi}(a u_\lambda - u_\lambda a u_\lambda)| \\ &\leq \phi(a^*a)^{1/2}\phi((1-u_\lambda)^2)^{1/2} + \phi(u_\lambda a^* a u_\lambda)^{1/2}\phi((1-u_\lambda)^2)^{1/2} \\ &\leq \phi((1-u_\lambda))^{1/2}(\phi(a^*a)^{1/2} + \phi(u_\lambda a^* a u_\lambda)^{1/2}) \\ &= 0. \end{aligned}$$

∎

5.10 PROPOSITION: *Let A be a nonzero* C*-*algebra and let $a \in A$ be a normal element. Then there is a state $\phi \in S(A)$ such that $\phi(a) = \|a\|$.*

PROOF: Let $B \subset \tilde{A}$ be the C*-subalgebra generated by a and 1. Since B is commutative, we have $\hat{a} \in C(\Omega(B))$ and $\phi \in \Omega(B)$ such that $\phi(a) = \hat{a}(\phi) = \|a\|$. Since $\phi(1) = 1$, there is a positive extension to $\tilde{A}$. Then the restriction ϕ to A satisfies the requirements, since $\|\phi\| = \phi(1) = 1$. ∎

5.11 A *representation* of a C*-algebra A is a (H, π) consisting of a Hilbert space H and a *-homomorphism $\pi : A \to B(H)$. If π is injective then we say (H, π) is *faithful*. If A has a faithful representation, then it is *-isomorphic to a closed self-adjoint subalgebra of $B(H)$, that is, a concrete C*-algebra.

5.12 We begin by establishing that every C*-algebra has many representations. Let A be a C*-algebra. Given a positive linear functional ϕ, let

$$N_\phi := \{a \in A \mid \phi(a^*a) = 0\},$$

which is a closed left ideal in A.

Define a map $\langle \cdot, \cdot, \rangle_\phi : A/N_\phi \times A/N_\phi \to \mathbb{C} \to \mathbb{C}$ by $\langle a + N_\phi, b + N_\phi \rangle \to \phi(b^*a)$.

EXERCISE: Show that $\langle \cdot, \cdot, \rangle_\phi$ defines an inner product on A/N_ϕ.

5.13 Let H_ϕ denote Hilbert space obtained by completing A/N_ϕ with respect to the inner product described above. For $a \in A$, we define a linear operator $\pi_\phi(a) : A \to \mathcal{B}(A/N_\phi)$ by $\pi_\phi(a)(b) = ab + N_\phi$. It is a straightforward exercise to check that this is a bounded *-homomorphism and hence extends to a *-homomorphism

$$\pi_\phi : A \to B(H_\phi).$$

5.14 DEFINITION: Let ϕ be a positive linear functional on the C*-algebra A. The representation (H_ϕ, π_ϕ) is called the Gelfand–Naimark–Segal representation associated to ϕ, or more commonly the *GNS representation* associated to ϕ.

5.15 Let $(H_\lambda, \pi_\lambda)_{\lambda \in \Lambda}$ be a family of representations of the C*-algebra A. Define

$$\oplus_{\lambda \in \Lambda} \pi_\lambda : A \to \oplus_{\lambda \in \Lambda} H_\lambda$$

to be the map taking $a \in A$ to the element with $\pi_\lambda(a)$ in the λ coordinate. Then $(\oplus_{\lambda\in\Lambda} H_\lambda, \oplus_{\lambda\in\Lambda}\pi_\lambda)$ is a representation of A. It is faithful as long as, for each $a \in A \setminus \{0\}$, there is some λ such that $\pi_\lambda(a) \neq 0$.

5.16 Definition: Let A be a C*-algebra. The representation

$$(\oplus_{\phi\in S(A)} H_\phi, \oplus_{\phi\in S(A)}\pi_\phi)$$

is called the *universal representation* of A.

5.17 From the GNS construction above, we get the following Gelfand–Naimark theorem.

Theorem: *Let A be a C*-algebra. Then its universal representation is faithful.*

Proof: Exercise. ▮

5.18 If $A \subset \mathcal{B}(H)$ is a C*-subalgebra, then its *commutant* is defined as

$$A' := \{b \in \mathcal{B}(H) \mid ab = ba \text{ for every } a \in A\}.$$

Clearly we have that $A \subset A''$. If in fact A contains $1_{\mathcal{B}(H)}$ and $A = A''$, then A is called a *von Neumann algebra*. Since $(A'')'' = A''$ for any C*-algebra $A \subset \mathcal{B}(H)$, A'' is itself a von Neumann algebra. We call A'' the *enveloping von Neumann algebra* of A, or the *double commutant* of A.

5.19 Since every C*-algebra is A contained in a von Neumann algebra A'', we may think of von Neumann algebras as "bigger" in a certain sense. That $A \subset A''$ is rather algebraic in nature, but this can also be seen topologically. In addition to the norm topology on $\mathcal{B}(H)$, we also have two weaker topologies: the weak operator and strong operator topologies.

Definition: The *weak operator topology* on $\mathcal{B}(H)$ is the weakest topology such that the sets $W(a, \xi, \mu) = \{b \in \mathcal{B}(H) \mid |\langle a\xi, \mu\rangle| < 1\}$ are open. The *strong operator topology* is the weakest topology in which the sets $S(a, \xi) = \{b \in \mathcal{B}(H) \mid \|(a - b)\xi\| < 1\}$ are open.

The sets $W(a_i, \xi_i, \mu_i \mid 1 \leq i \leq n) = \cap_1^n W(a_i, \xi_i, \mu_i)$ form a base for the weak operator topology and the similarly, sets of the form $W(a_i, \xi_i \mid 1 \leq i \leq n) = \cap_i^n S(a_i, \xi_i)$ are a base for the strong operator topology.

5.20 Proposition: *Let $(a_\lambda)_{\lambda\in\Lambda}$ be a net in $\mathcal{B}(H)$ and let $a \in \mathcal{B}(H)$. Then*

(i) *$(a_\lambda)_\Lambda$ converges to a in the weak operator topology, written $a_\lambda \xrightarrow{\text{WOT}} a$, if $\lim_\lambda \langle a_\lambda \xi, \eta\rangle = \langle a\xi, \eta\rangle$ for all $\xi, \eta \in H$;*

(ii) *$(a_\lambda)_\Lambda$ converges to a in the strong operator topology, written $a_\lambda \xrightarrow{\text{SOT}} a$, if $\lim_\lambda a_\lambda \xi = a\xi$ for all $\xi \in H$.*

Proof: Exercise. ▮

5.21 Exercise: Show that convergence in the norm operator topology implies convergence in the strong topology implies convergence in the weak topology. Show that the reverse implications do not neccessarily hold.

5.22 Proposition: *Let $a \in \mathcal{B}(H)$. Left and right multiplication by a is both SOT- and WOT-continuous.*

Proof: Suppose that $(b_\lambda)_\Lambda$ is a net in $\mathcal{B}(H)$ which WOT-converges to b. Let $\xi, \mu \in H$. Then

$$\begin{aligned}\lim_\lambda \langle ab_\lambda \xi, \mu\rangle &= \lim_\lambda \langle b_\lambda \xi, a^*\mu\rangle \\ &= \langle b\xi, a^*\mu\rangle \\ &= \langle ab\xi, \mu\rangle,\end{aligned}$$

showing $ab_\lambda \xrightarrow{\text{WOT}} ab$. Thus multiplication on the left is WOT-continuous. The other calculations are similar and are left as exercises. ∎

5.23 Proposition: *Let S be a subset of $\mathcal{B}(H)$. Then the commutant $S' = \{a \in \mathcal{B}(H) \mid ax = xa \text{ for all } x \in S\}$ is closed in the weak operator topology.*

5.24 Definition: If $A \subset \mathcal{B}(H)$ then we say that A acts nondegenerately on H if its null space

$$N_A := \{\xi \in H \mid a\xi = 0 \text{ for every } a \in A\}$$

is trivial.

5.25 Now we can show von Neumann's Double Commutant Theorem.

Theorem: *Let A be a C*-algebra acting nondegenerately on H. The following are equivalent.*

(i) *A is a von Neumann algebra;*
(ii) $\overline{A}^{\text{WOT}} = A$;
(iii) $\overline{A}^{\text{SOT}} = A$.

Proof: We have that $\overline{A}^{\text{SOT}} \subset \overline{A}^{\text{WOT}}$ since the strong operator topology is stronger than the weak topology. Since $A \subset A''$ and A'' is WOT-closed, we furthermore have $\overline{A}^{\text{WOT}} \subset A''$. It only remains to show that $A'' \subset \overline{A}^{\text{SOT}}$. Let $a \in A''$. It is enough to show that for any $n \in \mathbb{N} \setminus \{0\}$ and $\xi_1, \dots, \xi_n \in H$ there is some $b \in A$ such that b is contained in $S(a, \xi_i \mid 1 \le i \le n)$. Notice that this is equivalent to finding $b \in A$ with $\sum_{i=1}^n \|(b-a)\xi_i\|^2 < 1$.

Consider $n = 1$. Let p be the orthogonal projection onto $\overline{A\xi_1}$. Then if $c \in A$ we have $pcp\xi_1 = cp\xi_1$ and if $\mu \neq \xi$ then $pcp\mu = cp\mu = 0$. Thus $pcp = pc$ for every $c \in A$. Thus $cp = (pc^*)^* = (pc^*p)^* = pcp = pc$ whence $p \in A'$. Note that this means $p^\perp \in A'$, too.

If $\mu = p^{\perp}\xi_1$ then $A\mu = Ap^{\perp}\xi_1 = p^{\perp}A\xi_1 = 0$. Since A acts nondegenerately, we must therefore have $\mu = 0$. It follows that $\xi_1 \in \overline{A\xi_1}$. Since $a \in A''$ we have that $pa = ap$ and $a\xi_1 \in \overline{A\xi_1}$. Thus we can find $b \in A$ which satisfies $\|(a-b)\xi_1\| < 1$.

Now suppose $n \geq 2$. Let $H^{(n)} := H \oplus \cdots \oplus H$ be the direct sum of n copies of H. An arbitrary operator in $\mathcal{B}(H^{(n)})$ then looks like an $n \times n$ matrix $(x_{ij})_{ij}$ with each entry $x_{ij} \in \mathcal{B}(H)$. For $c \in A$, let $c^{(n)} \in \mathcal{B}(H^{(n)})$ be defined as $c^{(n)}(\xi_1, \dots, \xi_n) := (c\xi_1, \dots, c\xi_n)$ and then set $A^{(n)} := \{c^{(n)} \mid c \in A\}$.

We claim that $(A^{(n)})'' = (A'')^{(n)}$ where $(A'')^{(n)}$ is defined analogously to the above. It is easy to see that $c = (c_{ij})_{ij} \in (A^{(n)})'$ if and only if each $c_{ij} \in A'$. Thus $A^{(n)}$ contains all the matrix units e_{ij} where e_{ij} is the matrix with 1 in the (i,j)-entry and zero elsewhere. It follows that any $c \in (A^{(n)})''$ must commute with every e_{ij}. The only way this is possible if all the diagonal entries of c are the same and the off-diagonal entries are zero, that is $c = c_{11}^{(n)}$ where $c_{11} \in A''$. It is clear that c commutes with each $b^{(n)}$ where $b \in A'$. Thus $c_{11} \in A''$ and $c \in (A'')^n$, proving the claim.

Now we apply the case for $n = 1$ to $a^{(n)} \in (A^n)''$ and $\xi = (\xi_1, \dots, \xi_n)$ to find $b \in A$ with

$$1 > \|(a^{(n)} - b^{(n)})\xi\|^2 = \sum_{i=1}^{n} \|(a-b)\xi_i\|^2.$$

Thus b is in $S(a, \xi_i \mid 1 \leq i \leq n)$. ▮

5.26 Proposition: *The weak operator continuous linear functionals and the strong operator continuous linear functionals $\phi : \mathcal{B}(H) \to \mathbb{C}$ coincide and are always of the form*

$$\phi(a) = \sum_{i=1}^{n} \langle a\xi_i, \eta_i \rangle$$

for some $n \in \mathbb{N}$ and $\xi_i, \eta_i \in H$, $1 \leq i \leq n$.

Proof: It is easy to see that anything of the form $\phi(a) = \sum_{i=1}^{n} \langle a\xi_i, \mu_i \rangle$ will be a WOT and hence SOT continuous linear functional. Suppose now that ϕ is a SOT-continuous positive linear functional. Then the set $U := \{a \in \mathcal{B}(H) \mid |\phi(a)| < 1\}$ is open in the strong operator topology and contains zero. Thus we can find a basic set as given in Definition 5.19 containing zero that is completely contained in U. That is to say, there are $\xi_1, \dots, \xi_n \in H$ such that

$$V := \{a \in \mathcal{B}(H) \mid \|a\xi_i\| < 1, 1 \leq i \leq n\} \subset \{a \in \mathcal{B}(H) \mid |\phi(a)| < 1\}.$$

If $a \in \mathcal{B}(H)$ satisfies $\sum_{i=1}^{N} \|a\xi_i\| < 1$ then clearly $a \in V$. Let $H^{(n)} = H \oplus \cdots \oplus H$ be the Hilbert space that is given by the direct sum of n copies of H. Set $\xi := (\xi_1, \dots, \xi_n)$ and let $\psi_\xi : \mathcal{B}(H) \to H^{(n)}$ be the map given by $\psi_\xi(a) = (a\xi_1, \dots, a\xi_n)$. Now define

$$F : \psi_\xi(\mathcal{B}(H)) \to \mathbb{C}$$

by $F(\psi_\xi(a)) = \phi(a)$. Note that if $\eta \in \psi_\xi(\mathcal{B}(H))$, then $\eta = (a\xi_1, \ldots, a\xi_n)$ for some $a \in A$ and thus if $\|\eta\| \leq 1$ we have $\sum_{i=1}^n \|a\xi_i\| \leq 1$. In that case $\phi(a) \leq 1$. Thus $\|F\| \leq 1$ and, by applying the Hahn–Banach theorem, there is an extension to a continuous linear functional $\tilde{F} : H^{(n)} \to \mathbb{C}$. By the Riesz representation theorem, there is $\eta = (\eta_1, \ldots, \eta_n) \in H^{(n)}$ such that

$$\tilde{F}(\nu) = \langle \nu, \eta \rangle_{H^{(n)}} = \sum_{i=1}^{n} \langle \nu_i, \eta_i \rangle_H,$$

and so

$$\phi(a) = \tilde{F}(\psi(a)) = \sum_{i=1}^{n} \langle a\xi_i, \eta_i \rangle,$$

and the result follows since this is also WOT continuous. ∎

5.27 Proposition: *Let $A \subset \mathcal{B}(H)$ be a C*-algebra. Suppoes that $K \subset H$ is an A-invariant subspace, that is $AK = \{a\xi \mid a \in A, \xi \in K\} \subset K$. Let $p \in \mathcal{B}(H)$ be the orthogonal projection onto K. Then $p \in A'$.*

Proof: Since K is invariant $ap\xi \in K$ for every $a \in A$ and every $\xi \in H$. Thus $pap\xi = ap\xi$ for every $a \in A$ and $\xi \in H$, so $pa = (a^*p)^* = (pa^*p)^* = pap = ap$. Hence $p \in A'$. ∎

5.28 Lemma: *Let $A \subset \mathcal{B}(H)$ be a C*-algebra and $I \subset A$ an ideal. Then the projection $p \in \mathcal{B}(H)$ onto the closed subspace IH is in A'.*

Proof: Since $A(IH) = (AI)H = AH$, the subspace IH is invariant. The rest follows from the previous proposition. ∎

5.29 The following is a reformulation of the Hahn–Banach separation theorem which we will require in the proof of Theorem 4.15.

Lemma: *Let V be a locally compact topological vector space and let $S \subset V$ be a closed convex subset. Then if $x \notin S$ the sets $\{x\}$ and S are strictly separated, that is, there is a continuous linear functional $\phi \in V^*$, $r \in \mathbb{R}$ and $\epsilon > 0$ such that*

$$\Re(\phi(x)) < r < r + \epsilon < \Re(\phi(s)) \text{ for every } s \in S.$$

5.30 We are now able to prove Theorem 4.15: that every ideal I in a C*-algebra A has an approximate unit which is quasicentral for A.

Proof of Theorem 4.15: Let A be a C*-algebra, $I \subset A$ an ideal and let $\pi : A \to \mathcal{B}(H)$ be the universal representation of A. Since (H, π) is faithful, it is enough to show the theorem holds for $\pi(A)$ and $\pi(I_)$.

Let $(u_\lambda)_\Lambda$ be an approximate unit for $\pi(I) \subset \pi(A)$. Let $\mathcal{E}$ denote the convex hull of $\{u_\lambda \mid \lambda \in \Lambda\}$, that is,

$$\mathcal{E} = \{\textstyle\sum_{i=1}^n \mu_i x_i \mid \sum_{i=1}^n \mu_i = 1 \text{ and } x_i = u_\lambda \text{ for some } \lambda \in \Lambda\}.$$

If $\sum_{i=1}^n \mu_i x_i \in \mathcal{E}$ then there is some u_λ such that $x_i \leq u_\lambda$ for every $1 \leq i \leq n$. Thus $\sum_{i=1}^n \mu_i x_i \leq \sum_{i=1}^n \mu_i u_\lambda = u_\lambda$, which shows that $\mathcal{E}$ is upwards directed and is furthermore an approximate unit for I.

We will show that for every $\lambda_0 \in \Lambda$ and finite set of elements $a_1, \dots, a_n \in \pi(A)$ there is $e \in \mathcal{E}$ $e \geq u_{\lambda_0}$ with

$$\|a_i e - e a_i\| < 1/n \text{ for every } 1 \leq i \leq n.$$

So let $a_1, \dots, a_n$ and u_{λ_0} be given. Let $H^{(n)}$ denote the Hilbert space given by the direct sum of n copies of H and for an element $a \in \pi(A)$, denote by $a^{(n)} \in M_n(\pi(A))$ the element with a copied n times down the diagonal. Notice that $(u_\lambda^{(n)})_\Lambda$ is an approximate unit for $M_n(\pi(I))$. Let $\mathcal{E}_{\lambda \geq \lambda_0}$ denote the convex hull of $\{u_\lambda \mid \lambda \geq \lambda_0\}$. Let $c = a_1 \oplus \cdots \oplus a_n$.

Set

$$\mathcal{S} := \{c e^{(n)} - e^{(n)} c \mid e \in \mathcal{E}_{\lambda \geq \lambda_0}\}.$$

We claim that $0 \in \overline{\mathcal{S}}$. To show this, suppose it is not. Since $\mathcal{E}_{\lambda \geq \lambda_0}$ is convex, it is easy to see that $\mathcal{S}$ is convex. Then by the Lemma 5.29 there is a continuous linear functional $\phi \in (\mathcal{B}(H))^*$, $r \in \mathbb{R}$ and $\epsilon > 0$ such that

$$0 = \Re(\phi(0)) < r < r + \epsilon < \Re(\phi(s))$$

for every $s \in \mathcal{S}$. Rescaling if necessary, we may assume that $1 \leq \Re(\phi(s))$ for every $s \in \mathcal{S}$. Thus there is $\xi, \eta \in H^{(n)}$ such that $\phi(a) = \langle a\xi, \eta \rangle$.

We have $u_\lambda \xrightarrow{\text{SOT}} p$ where p is the projection onto $\pi(I)H$. Since $p \in \pi(A)'$,

$$\begin{aligned} \phi(c u_\lambda^{(n)} - u_\lambda^{(n)} c) &= \langle (c u_\lambda^{(n)} - u_\lambda^{(n)} c)\xi, \eta \rangle \\ &\to \langle (c p^{(n)} - p^{(n)} c)\xi, \eta \rangle \\ &= 0, \end{aligned}$$

contradicting the fact that $\phi(s) \geq 1$ for every $s \in \mathcal{S}$. Thus we must have that $0 \in \mathcal{S}$, which proves the claim.

It follows that for any $n \in \mathbb{N} \setminus \{0\}$, finite subset $\mathcal{F} = \{a_1, \dots, a_n\}$ and $\lambda_0 \in \Lambda$ there is $f_{\mathcal{F},\lambda_0} \in \mathcal{E}_{\lambda \geq \lambda_0}$ such that

$$\|(a_1 \oplus \cdots \oplus a_n) f_{\mathcal{F},\lambda_0}^{(n)} - f_{\mathcal{F},\lambda_0}^{(n)} (a_1 \oplus \cdots \oplus a_n)\| < 1/n.$$

Thus

$$\begin{aligned} \|a_i f_{\mathcal{F},\lambda_0} - f_{\mathcal{F},\lambda_0} a_i\| &\leq \max_{1 \leq j \leq n} \|a_i f_{\mathcal{F},\lambda_0} - f_{\mathcal{F},\lambda_0} a_i\| \\ &= \|(a_1 \oplus \cdots \oplus a_n) f_{\mathcal{F},\lambda_0}^{(n)} - f_{\mathcal{F},\lambda_0}^{(n)} (a_1 \oplus \cdots \oplus a_n)\| \\ &< 1/n. \end{aligned}$$

It follows that $(f_{\mathcal{F},\lambda})_{\mathcal{F},\lambda}$, where $\mathcal{F}$ runs over all finite subsets of $\pi(A)$ and $\lambda \in \Lambda$, is a quasicentral unit approximate unit for $\pi(I)$.

Exercises

5.1 Show that any positive linear functional $\phi : A \to \mathbb{C}$ admits a unique extension $\tilde{\phi} : \tilde{A} \to \mathbb{C}$ such that $\|\tilde{\phi}\| = \|\phi\|$.

5.2 Let $A = M_n$ and $H = \mathbb{C}^m$ for $n, m \in \mathbb{N}$. For what values of m does A admit a faithful representation on H? Show that if $\pi_1, \pi_2 : M_n \to M_m$ are both faithful representations on $\mathbb{C}^m$, then they are unitarily equivalent: there exists a unitary $u \in M_m$ such that $\pi_1(a) = u^*\pi_2(1)u$ for every $a \in M_n$.

5.3 Prove that the universal representation of a C*-algebra A is always faithful. Thus abstract C*-algebras and concrete C*-algebras coincide.

5.4 Show that for any C*-algebra A, $M_n(A)$ is also a C*-algebra. Show that $M_n(A) \cong M_n \otimes A$, where $\otimes$ denotes the algebraic tensor product. (See for example [2].) In general, there is more than one way to take the tensor product of C*-algebras in such a way that their algebraic tensor product is a dense *-subalgebra. If A is a C*-algebra such that $A \otimes_1 B \cong A \otimes_2 B$ for any C*-algebra B and any C*-tensor product is called *nuclear*. Deduce that M_n is a nuclear C*-algebra.

5.5 Let H be a Hilbert space. In $\mathcal{B}(H)$, show that convergence in the operator norm topology implies strong operator convergence which in turn implies weak operator convergence.

5.6 Show that the commutant S' of a subset $S \subset \mathcal{B}(H)$ is closed in the weak operator topology. Show that if $S = S^*$ then S' is a *-algebra.

5.7 Let A be a C*-algebra and let (H, π) be the universal representation of A. Show that $\overline{\pi(A)}^{\mathrm{SOT}}$ is a von Neumann algebra.

5.8 Let $A \subset \mathcal{B}(H)$ be C^*-algebra with approximate unit $(u_\lambda)_\Lambda$. Does $(u_\lambda)_\Lambda$ converge in the strong operator or weak operator topology?

5.9 Let $(a_\lambda)_\Lambda \subset \mathcal{B}(H)$ be a net that is WOT-convergent. Show that $(a_\lambda)_\Lambda$ must be norm bounded.

5.10 Let B be a strongly closed hereditary subalgebra of A. Show that there is a unique projection $p \in B$ such that $B = pAp$.

6. Further examples of C*-algebras

UHF algebras and AF algebras. 6.1 A C*-*seminorm* on a $*$-algebra A is a seminorm p on A such that, for all a and b in A, we have $p(ab) \leq p(a)p(b)$, $p(a^*) = p(a)$ and $p(a^*a) = p(a)^2$.

Let $p : A \to \mathbb{R}_+$ be a C*-seminorm on a $*$-algebra A. Then $N = \ker(p)$ is a self-adjoint ideal in A, and this induces a C^*-norm on the quotient A/N given by $\|a + N\| = p(a)$. Let $B = \overline{A/N}^{\|\cdot\|}$ be the completion with respect to this norm. Define the multiplication and involution in the obvious way. This makes B into a C*-algebra called the *enveloping C^*-algebra* of (A, p).

The map $i : A \to B : a \to a + N$ is called the *canonical map* and the image of A under i is a dense $*$-subalgebra of B.

6.2 An *inductive sequence* of C*-algberas $(A_n, \phi_n)_{n\in\mathbb{N}}$ consists of a sequence of C*-algebras $(A_n)_{n\in\mathbb{N}}$ and a sequence of connecting *-homomorphisms $(\phi_n : A_n \to A_{n+1})_{n\in\mathbb{N}}$

6.3 Proposition: *Let $(A_n, \phi_n)_{n\in\mathbb{N}}$ be an inductive sequence of* C*-*algebras. Let* $\mathcal{A} = \{a = (a_j)_{j\in\mathbb{N}} \subset \Pi_{j\in\mathbb{N}} A_j \mid$ *there is $N \in \mathbb{N}$ such that $a_{j+1} = \phi_j(a_j)$ for all $j \geq N$*$\}$. *Then $\mathcal{A}$ is a *-algebra under pointwise operations and*

$$p : \mathcal{A} \to \mathbb{R}_+ : a \mapsto \lim_{k\to\infty} \|a_k\|_{A_k}$$

is a C^*-seminorm on $\mathcal{A}$.

Proof: Exercise. ∎

6.4 Definition: Let $(A_n, \phi_n)_{n\in\mathbb{N}}$ be an inductive sequence of C*-algebras. The *inductive limit* of $(A_n, \phi_n)_{n\in\mathbb{N}}$, written $\varinjlim(A_n, \phi_n)$, (or simply $\varinjlim A_n$ if it's clear what the maps should be) is the enveloping C*-algebra of $(\mathcal{A}, p)$, where $\mathcal{A}$ and p are the *-algebra and C*-seminorm, respectively, as defined in Proposition 6.3.

6.5 Let $(A_n, \phi_n)_{n\in\mathbb{N}}$ be an inductive sequence of C*-algebras and let $A = \varinjlim A_n$ be the inductive limit. It is useful to describe maps between nonadjacent C*-algebras in the inductive sequence, as well as from each A_n in the sequence to the A. Thus we define, for $n < m$

$$\phi_{n,m} : A_n \to A_m$$

to be the composition

$$\phi_{m-1} \circ \cdots \circ \phi_{n+1} \circ \phi_n, .$$

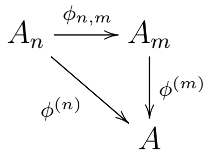
If $a \in A_n$, then define $(a_j)_{j\in\mathbb{N}} \subset \Pi_{j\in\mathbb{N}} A_j$ by

$$a_j = \begin{cases} 0 & \text{if } j < n, \\ a & \text{if } j = n, \\ \phi_{n,j-1}(a) & \text{if } j > n \end{cases}$$

Clearly $(a_j)_{j\in\mathbb{N}} \in \mathcal{A}$. From this we define the map

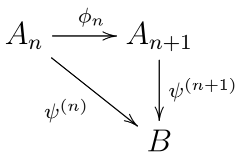
$$\phi^{(n)} : A_n \to A$$

by $\phi^{(n)}(a) = \iota((a_j)_{j\in\mathbb{N}}$ where $\iota : \mathcal{A} \to A/N_p$ is the canonical map from $\mathcal{A}$ into its enveloping C*-algebra A. From this we get, for every n, m $n < m$, a commutative diagram

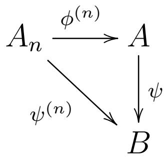


This leads to the following universal property.

6.6 THEOREM: *Let $(A_n, \phi_n)_{n\in\mathbb{N}}$ be an inductive sequence of* C*-*algebras with limit* $A = \varinjlim A_n$. *Suppose there is a* C*-*algebra B and for every $n \in \mathbb{N}$ there are* *-*hommoorphisms $\psi^{(n)} : A_n \to B$ making the diagrams*



commute. Then there is a unique *-*homomorphism $\psi : A \to B$ making the diagrams*



commute.

PROOF: Let B and the *-homomorphisms $\psi^{(n)} : A_n \to B$ be given. If $(a_j)_{j\in\mathbb{N}} \in \mathcal{A}$, then there is $N \in \mathbb{N}$ such that for every $j \geq N$ we have $a_{j+1} = \phi_j(a_j)$. By commutativity of the first diagram we have that $\psi^N(a_N) = \psi^j(a_j)$ for every $j \geq N$.

Suppose that $a \in A_n$, $b \in A_m$ and $\phi^{(n)}(a) = \phi^{(m)}(b) \in \varinjlim A_n$. If $n \leq m$ then $\phi^{(m)} \circ \phi_{n,m}(a) = \phi^{(m)}(b)$ by commutativitiy of the diagram in (6.5). It follows that $\lim_{k\to\infty, k\geq m} \|\phi_{n,k}(a) - \phi_{m,k}(b)\| = 0$. Thus $\lim_{k\to\infty} \|\psi^k(\phi_{n,k}(a)) - \psi^k(\phi_{m,k}(b))\| = 0$.

The above shows that $\psi : \iota(\mathcal{A}) \to B$ defined on each $(a_j)_{j\in\mathbb{N}}$ by $\iota \circ \psi^N(a_N)$ for sufficiently large N, is well-defined and extends to a *-homomorphism $\psi : \varinjlim A \to B$, making the second diagram commute. ∎

6.7 EXERCISE: Let $A = \varinjlim(A_n, \phi_n)$ be the inductive limit of a sequence of C*-algebras and let B be a C*-algebra. For every $n \in \mathbb{N}$ let $\psi^{(n)} : A_n \to B$ be a *-homomorphism satisfying $\psi^{(n+1)} \circ \phi_n = \psi^{(n)}$ and let $\psi : A \to B$ be the induced *-homomorphism. Then

(i) ψ is injective if and only if $\ker(\psi^{(n)}) \subset \ker(\phi^{(n)})$ for every $n \in \mathbb{N}$, and
(ii) ψ is surjective if and only if $B = \overline{\cup_{j=1}^{\infty} \psi^{(n)}(A_n)}$.

6.8 A *supernatural number* $\mathfrak{p}$ is given by the infinite product $\mathfrak{p} = \Pi_{p \text{ prime}} p^{k_p}$ where $k_p \in \mathbb{N} \cup \{\infty\}$. Every natural number is thus a supernatural number. A supernatural number is of *infinite type* if, for every prime p, we have either $k_p = 0$ or $k_p = \infty$.

6.9 To every supernatural number $\mathfrak{p}$, we may associate an inductive system of matrix algebras as follows. We may choose natural numbers $(n_i)_{i \in \mathbb{N}}$ such that, for each i, n_i divides n_{i+1} and n_i divides $\mathfrak{p}$, but n_i does not divide $\mathfrak{p}p$ for any prime p unless p^∞ divides $\mathfrak{p}$. Call such a sequence a UHF decomposition of $\mathfrak{p}$. For each n_i, let $\phi_i : M_{n_i} \to M_{n_{i+1}}$ be the map that sends a matrix a of size n_i to the block matrix of size n_{i+1} by copying a n_{i+1}/n_i times down the diagonal:

$$\phi : M_{n_i} \to M_{n_{i+1}} : a \mapsto \left(\begin{array}{cccc} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{array} \right) \left. \vphantom{\begin{array}{c} a \\ 0 \\ \vdots \\ 0 \end{array}} \right\} \frac{n_{i+1}}{n_i} \text{ times.}$$

6.10 Definition: Let $\mathfrak{p}$ be a supernatural number and $(n_k)_k \subset \mathbb{N}$ a UHF sequence for $\mathfrak{p}$. The *uniformly hyperfinite* (UHF) algebra of type $\mathfrak{p}$ is the inductive limit of the inductive system (M_{n_i}, ϕ_i),

$$\mathcal{U}_\mathfrak{p} = \varinjlim M_{n_i}.$$

6.11 Exercise: Show that the above is well-defined.

6.12 Theorem: *Let (A_n, ϕ_n) be an inductive limit of simple* C*-*algebras. Then $\varinjlim A_n$ is simple.*

Proof: A C*-algebra A is simple if and only whenever B is another C*-algebra and $\psi : A \to B$ is a surjective *-homomorphism, then ϕ is injective. Suppose then that $\psi : \varinjlim A_n \to B$ is a surjection onto a nonzero C*-algebra B. For any $n \in \mathbb{N}$, $\phi^{(n)}(A_n) \subset A$ is the image of a simple C*-algebra and so is also simple. Thus $\psi|_{\phi^{(n)}(A_n)} : \phi^{(n)}(A_n) \to B$ is either zero or injective. Since $A \cong \overline{\cup_{n \in \mathbb{N}} \phi^{(n)}(A)}$ and $\cup_{n \in \mathbb{N}} \phi^{(n)}(A)$ is dense, there must be some $N \in \mathbb{N}$ and $a \in \phi^{(N)}(A)$ such that $\psi(a) \neq 0$. In this case $\psi|_{\phi^{(N)}(A_N)}$ must be injective.

Now, if $k < N$ then $\phi^{(k)}(A_k) \subset \phi^{(n)}(A_n)$. Thus $\psi|_{\phi^{(k)}(A_k)}$ is nonzero and hence injective. If $k > N$ then $\phi^N(A_N) \subset \phi^k(A_k)$. Then $a \in \phi^k(A_k)$ so again $\psi|_{\phi^{(k)}(A_k)}$ is nonzero and hence injective. Thus $\psi : A \to B$ is injective on a dense subset, hence injective. ∎

6.13 Corollary: *A UHF algebra is simple.*

6.14 An approximately finite-dimensional (AF) algebra generalises UHF algebras. Now in the inductive sequence, we allow any finite C*-subalgebra F to appear, not only matrix algebras.

DEFINITION: An approximately finite-dimensional (AF) algebra is the inductive limit of a sequence (F_n, ϕ_n) where F_n is a finite dimensional C*-algebra for every $n \in \mathbb{N}$.

6.15 Note that a UHF algebra is an AF algebra, but the opposite need not be the case. For example, the compact operators $\mathcal{K}$ is an AF algebra but is not even unital. Still, AF algebras are a very nice class of C*-algebras and are quite tractable: we can say a lot about their structure. We saw that a UHF algebra is uniquely determined by its associated supernatural number. For AF algebras, the situation is slightly more complicated. Instead of a single supernatural number, we require a so-called dimension group to distinguish AF algebras. In the simple unital case, this boils down to the pointed, ordered K_0-group: the Grothendieck group of Murray–von Neumann equivalence classes of projections in matrix algebras over the AF algebra, together with the order on K_0 induced by the order on positive elements, as well as the class of the unit. Since K_0 respects inductive limits, this is an easily computable invariant for AF algebras. We'll have more to say on this later on, in the meantime, we have another nice structural property of AF algebras.

6.16 Recall that a partial isometry v in a C*-algebra A is an element such that both v^*v and vv^* are projections.

DEFINITION: Let A be a C*-algebra. If $p, q \in A$ are projections, then p is *Murray–von Neumann equivalent* to q if there is a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* = q$. The projection p is *Murray–von Neumann subequivalent* to q if there is a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* \leq q$.

A projection $p \in A$ is called *finite* if p is not Murray–von Neumann equivalent to a proper subprojection of itself, that is, there is no $v \in A$ with $v^*v = p$ and $vv^* \leq p$ but $vv^* \neq p$; otherwise p is said to be *infinite*.

A unital C*-algebra A is called *finite* if 1_A is finite. A unital C*-algebra A is called *stably finite* if $M_n(A)$ is finite for every $n \in \mathbb{N}$.

6.17 An *isometry* in a unital C*-algebra A is an element $s \in A$ with $s^*s = 1_A$. Clearly any unitary in A is an isometry, but an isometry need not be a unitary in general. (See exercises.) If, however, this is the case, then A is finite:

PROPOSITION: *Let A be a unital* C*-*algebra. Suppose that every isometry in A is a unitary. Then A is finite.*

PROOF: Suppose that there is $p \in A$ with $p \leq 1_A$ and $v \in A$ such that $v^*v = 1_A$ and $vv^* = p$. But $v^*v = 1_A$ means that v is an isometry and hence unitary. Thus we have $p = vv^* = 1_A$ which shows that 1_A is finite. ∎

6.18 THEOREM: *If A is a unital AF algebra, then A is stably finite.*

PROOF: Exercise. ∎

6.19 As has been regularly mentioned, because of the fact that the Gelfand transform is a *-isomorphism, C*-algebras are often thought of "noncommutative" locally compact Hausdorff spaces. In the topological setting, there are several notions of the topological dimension of a space. A point should be zero dimensional, an interval one-dimensional, an n-cube n-dimensional, and so forth.

In what follows, we will denote the indicator function of an open set U by χ_U; thus

$$\chi_U(x) = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases}$$

DEFINITION: Let X be a locally compact Hausdorff space. We say that X has *covering dimension* d, written $\dim(X) = d$ if d is the least integer such that the following holds: For every open cover $\mathcal{O}$ of X there is a finite refinement $\mathcal{O}'$ such that, for every $x \in X$, $\sum_{U\mathcal{O}'} \chi_U(x) \leq d+1$. If no such d exists, we say that $\dim(X) = \infty$.

In the case that we restrict ourselves to locally compact metrisable spaces, the various definitions coincide with the covering dimension.

Moving to the noncommutative setting, we would like to find an analogue of the dimension of a space. At the commutative level, metrisable corresponds to the C*-algebra being separable, so we content ourselves with trying to establish noncommutative versions of covering dimension for separable C*-algebras. These should extend covering dimension in the sense that the noncommutative dimension of $C_0(X)$ should be the same as the covering dimension. There are a few such extensions; we describe three of them below.

6.20 DEFINITION: Let A be a unital separable C*-algebra A. We say that A has *real rank* d, written $RR(A) = d$, if d is the least integer such that, whenever $0 \leq n \leq d+1$ the following holds: For every n-tuple $(a_1, \ldots, a_n)$ of self-adjoint elements in A and every $\epsilon > 0$ there exists an n-tuple $(y_1, \ldots, y_n) \subset A_{sa}$ such that $\sum_{k=1}^n y_k^* y_k$ is invertible and $\| \sum_{k=1}^n (x_k - y_k)^*(x_k - y_k)\| < \epsilon$. If there is no such d, then we say the real rank of A is infinite.

6.21 The stable rank has a very similar definition, dropping the fact that the n-tuples need be self-adjoint, and we only look at n-tuples from $0 \leq n \leq d$ (rather than $d+1$).

DEFINITION: Let A be a unital separable C*-algebra A. We say that A has *stable rank* d, writen $SR(A) = d$ if d is the least integer such that, whenever $1 \leq n \leq d$ the following holds: For every n-tuple $(a_1, \ldots, a_n)$ of elements in A and every $\epsilon > 0$ there exists an n-tuple $(y_1, \ldots, y_n) \subset A$ such that $\sum_{k=1}^n y_k^* y_k$ is invertible and $\| \sum_{k=1}^{n} (x_k - y_k)^*(x_k - y_k)\| < \epsilon$. If there is no such d, then we say the real rank of A is infinite.

6.22 The nuclear dimension has a different flavour to the real and stable rank. It is a refinement of the completely positive approximation property, which, for C*-algebras, is equivalent to nuclearity (that is, A has the completely positive approximation property if and only if A is nuclear).

DEFINITION: Let A be a separable C*-algebra. We say that A has nuclear dimension d, written $\dim_{\mathrm{nuc}} A = d$, if d is the least integer satisiying the following: For every finite subset $\mathcal{F} \subset A$ and every $\epsilon > 0$ there are a finite dimensional, C*-algebra with $d+1$ ideals, $F = F_0 \oplus F_d$, and completely positive maps $\psi : A \to F$ and $\phi = F \to A$ such that ψ is contractive, $\phi|_{F_n}$ is contractive and orthogonality preserving (ie. for any $a, b \in (F_n)_+$ with $ab = ba = 0$, we have $\phi(a)\phi(b) = 0$) and

$$\|\phi \circ \psi(a) - a\| < \epsilon \text{ for every } a \in \mathcal{F}.$$

If no such d exists, then we say $\dim_{\mathrm{nuc}}(A) = \infty$.

A completely positive contractive map which is orthogonality preserving (such as $\phi|_{F_n}$ of the previous definition) is called *order zero.*

6.23 We should think of AF algebras of "zero-dimensional" objects. Note, however, that there is no definition for "stable rank zero". We get the following theorem for AF algebras. The proof is an exercise; see the exercise sheet for a bit of a hint.

THEOREM: *Let A be a unital AF algebra. Then A has real rank zero, stable rank one and nuclear dimension zero.*

If A is nonunital, we can define its real and stable rank by putting $RR(A) := RR(\tilde{A})$ and $SR(A) := SR(\tilde{A})$. Then the above theorem is also true for nonunital AF algebras.

Group C*-algebras. 6.24 Now that we know a bit about representations and the different topologies on $\mathcal{B}(H)$ we can construct C*-algebras out of locally compact groups. We will also assume throughout this section that the groups are Hausdorff. A topological group is a group together with a topology which makes group operations continuous. In particular, any group is a topological group with the discrete topology.

6.25 A Borel measure on a group G is left-translation-invariant if, for any Borel set $E \subset G$ and any $s \in G$, we have $\mu_G(sE) = \mu_G(E)$.

THEOREM: *Let G be a locally compact group. There is a left-translation-invariant Borel measure on G, denoted μ_G, which is unique up to scalar multiple.*

This measure is called a (left) *Haar measure* of G. When G is compact, $\mu(G)$ is finite and so we can normalise it so that $\mu(G) = 1$. If G is infinite and discrete, then we normalise so that $\mu(\{e\}) = 1$.

6.26 In general, a left translation-invariant measure need not be right-translation-invariant. However, if this is the case, then we call G unimodular. Unimodular groups include the cases that G is abelian, discrete, or compact. For the sake of brevity, we will stick to the unimodular case, though most of what we'll do can be generalised.

6.27 There are a number of noncommutative algebras that we can associate with a group. The first is the *group algebra* of G, which is the algebra of formal $\mathbb{C}$-linear combinations and is denoted ℂG. The multiplication in $\mathbb{C}G$ extends the group multiplication.

6.28 We also have the function algebra of compactly supported functions on G, denoted $C_c(G)$, which comes equipped with convolution as multiplication,

$$f * g(t) = \int_G f(s)g(s^{-1}t)d\mu(s),$$

and inversion for involution,

$$f^*(s) = \overline{f^*(s^{-1})}.$$

There is also a norm on $C_c(G)$ given by $\|f\|_1 = \int_G |f(t)|d\mu(t)$.

Notice that $\mathbb{C}G \subset C_c(G)$, but they are not the same unless G is discrete. When G is locally compact, $C_c(G)$ will not be complete with respect to this norm. Completing with respect to $\|\cdot\|_1$ gives us yet another group algebra, $L_1(G, \mu)$.

6.29 The space $L^1(G, \mu)$ consists of functions $f : G \to \mathbb{C}$ such that

$$\int_G |f(t)|d\mu(t) < \infty.$$

It is a Banach *-algebra when equipped with $\|\cdot\|_1$, convolution and inversion as in the case of $C_c(G)$.

6.30 We have $\mathbb{C}G \subset C_c(G) \subset L^1(G, \mu)$. $C_c(G)$ is a dense *-subalgebra of $L^1(G)$. If G is finite, then $L^1(G, \mu) \cong \mathbb{C}G$. Since a left Haar measure is unique up to a scalar multiple, for any two left Haar measures μ, μ' we have $L^1(G, \mu) \cong L^1(G, \mu')$. Thus henceforth we will write $L^1(G)$ for any $L^1(G, \mu)$ defined with respect to a left Haar measure.

The $\|\cdot\|_1$ norm is not a C*-norm in general, so $L^1(G)$ is not a C*-algebra. Thus we would like to find a *-homomorphism from $L^1(G)$ into a C*-algebra so that we can complete the image of $L^1(G)$ to a C*-algebra.

6.31 Proposition: *Let G be a locally compact group. Then $L^1(G)$ is unital if and only if G is discrete. In any case, $L^1(G)$ always has a norm one approximate unit.*

Proof: If G is discrete, then a unit is given by the function that is 1 and the identity $e \in G$ and zero everywhere else. If $L^1(G)$ is unital then $1 * f(e) =$

$\int_G 1(s)f(s^{-1})d\mu(s) = f(e)$ for every $f \in L^1(G,\mu)$ only if $1(s) = 0$ for every $s \neq e$. But then if G is not discrete $1 = 0$ a.e.μ, thus is not a unit in $L^1(G,\mu)$.

For any G, let $\mathcal{O}$ be the collection of open neighbourhoods E of e (the identity in G). For $E \in \mathcal{O}$, let f_E be a function in $L^1(G)$ with $f(e) = 1$, $\mathrm{supp}(f_E) \subset E$, $f_E^* = f^*$ and $\|f\|_1 = 1$. Since the set of all such neighbourhoods is upwards directed with respect to reverse containment, $(f_E)_E$ is an approximate unit for $L^1(G)$. ▌

6.32 Definition: Let G be a locally compact group. A *unitary representation* of G is given by a pair (H, u) consisting of a Hilbert space H and a strongly continuous homomorphism $u : G \to \mathcal{U}(H)$, where $\mathcal{U}(H)$ is the group of unitary operators on H. Here, strongly continuous means that $g \mapsto u(g)\xi$ is continuous for every $\xi \in H$.

We say that a unitary representation is irreducible if $u(G)$ does not commute with any proper projections in $\mathcal{B}(H)$.

6.33 Let $A \subset \mathcal{B}(H)$ be a C*-algebra. A is said to be *irreducible* whenever $K \subset H$ is closed vector subspace with $AK \subset K$, then $K \in \{0, H\}$.

Proposition: *Let G be a locally compact group and $u : G \to \mathcal{U}(H)$ a unitary representation. Then the* C*-*subalgebra of $\mathcal{B}(H)$ generated by $u(G)$, written* C*$(u(G))$, *is irreducible if and only if $u(G)$ is irreducible.*

Proof: Exercise. ▌

6.34 A representation (H, π) of A is called *nondegenerate* if the linear span of $\{\pi(a)\xi \mid a \in A, \xi \in H\}$, denoted by $\pi(A)(H)$, is dense in H, or, equivalently, for each $\xi \in H \setminus \{0\}$ there is $a \in A$ such that $\pi(a)(\xi) \neq 0$.

If $u : G \to \mathcal{U}(H)$ is a unitary representation then $\pi : L^1(G) \to \mathcal{B}(H)$ given by

$$\pi(f)\xi = \int_G f(t)u_t(\xi)d\mu(t), \text{ for } f \in L^1(G), \xi \in H$$

is a representation of $L^1(G)$. (Here we write u_t to denote the operator $u(t)$.)

Conversely, if we have a representation $\pi : L^1(G) \to \mathcal{B}(H)$ which is nondegenerate then we can find a unique unitary representation of G as follows: Then we have

$$\lim_{E \in \mathcal{O}} \pi(f_E)\pi(g)\xi = \pi(g)\xi$$

for every $g \in L^1(G)$. It follows that $\pi(f_E) \xrightarrow{\text{SOT}} 1_{\mathcal{B}(H)}$.

Define $u : G \to \mathcal{B}(H)$ by

$$u(s)\pi(g)\xi = \pi(g_s)\xi \text{ for } s \in G, \xi \in H$$

where $g_s(t) = g(s^{-1}t)$. In this case we have that $u(s) = \mathrm{SOT}\lim_{E \in \mathcal{O}} \pi((f_E)_s)$, which in turn implies that u is contractive. Furthermore, it is not hard to check that $u(s)$ is unitary for every $s \in G$. The construction ensures that this u is unique.

6.35 Definition: Let G be a locally compact group with Haar measure μ. The *left regular representation* of G on the Hilbert space $L^2(G)$, $\lambda : G \to \mathcal{U}(L^2(G))$, is given by

$$\lambda(s)f(t) = f(s^{-1}t).$$

(Check that this is indeed a unitary representation of G.)

6.36 Definition: The *reduced group* C^**-algebra* of G, written $\mathrm{C}^*_r(G)$ is the closure of $\lambda(L^1(G))$ in $\mathcal{B}(L^2(G))$. The *full group* C^**-algebra*, denoted $\mathrm{C}^*(G)$ is the closure of $L^1(G)$ under the direct sum of all irreducible representations, or, equivalently, $\mathrm{C}^*(G)$ is the completion of $L^1(G)$ with respect to the norm

$$\|f\| = \sup\{\|\pi(f)\| \mid \pi : L^1(G,\mu) \to \mathcal{B}(H) \text{ is a } {}^*\text{-representation}\}.$$

6.37 Definition: Given a locally compact abelian group G, a *character* of G is a continuous group homomorphism from $G \to \mathbb{T}$. The set of all characters of G has the structure of a compact abelian group, which we call the Pontryagin dual of G and denote by $\hat{G}$.

6.38 Let $f \in L^1(G)$. The *Fourier–Plancheral transform* $\hat{f}$ on $\hat{G}$ is given by

$$\hat{f}(\gamma) = \int_G \overline{\gamma(t)} f(t) d\mu(t).$$

Exercise: For $f, g \in L^1(G)$ we have $(f * g)\hat{} = \hat{f}\hat{g}$ and $(f^*)\hat{} = \overline{\hat{f}}$.

6.39 We will require the following, which we use without proof in Theorem 6.40

Theorem: *[Plancheral Theorem] The Fourier–Plancheral transform extends from a map $L^1(G) \to L^1(\hat{G})$ to a unitary operator from $L^2(G) \to L^2(\hat{G})$.*

6.40 Theorem: *Let G be an abelian group. Then $\mathrm{C}^*(G) \cong \mathrm{C}^*_r(G) \cong C_0(\hat{G})$.*

Proof: Let $f, g \in L^1(G)$. Then $f * g(t) = \int_G f(s)g(s^{-1}t)d\mu(s)$ and putting $x = s^{-1}t$, we get $s = tx^{-1} = x^{-1}t$ so

$$\int_G f(s)g(s^{-1}t)d\mu(s) = \int_G f(x^{-1}t)g(x)d\mu(x) = \int_G g(x)f(x^{-1}t)d\mu(x),$$

and we have $f * g = g * f$, that is, $L^1(G)$ is commutative.

Let $\Gamma : L^1(G) \to C_0(\Omega(L^1(G))$ be the Gelfand transform. Recall that $\Omega(L^1(G))$ is the character space of $L^1(G)$ and notice that a character is exactly a one-dimensional representation. As we saw above, every representation of $L^1(G)$ corresponds to a unitary representation of G on the same Hilbert space; here the Hilbert space is $\mathbb{C}$. The one-dimensional unitary representations of G are just the characters of G, that is, $\hat{G}$. Thus the Gelfand transform maps $L^1(G) \to C_0(\hat{G})$. Moreover, we have

$$f \mapsto \hat{f}$$

where $\hat{f}$ is the Fourier–Plancheral transform of f. The range of $\Gamma(L^1(G))$ is clearly self-adjoint. Moreover, is separates points by definition of the Fourier–Plancheral transform. Thus $\Gamma(L^1(G))$ is dense in $C_0(\hat{G})$. By Plancheral's Theorem, this extends to a unitary operator $u : L^2(G) \to L^2(\hat{G})$. Then,

$$u(\lambda(f))u^*(\hat{g}) = u(\lambda(f))g = (f * g)\hat{} = \hat{f}\hat{g},$$

when $f \in L^1(G)$ and $g \in L^2(G) \cap L^1(G)$. Thus $C_0(G) \ni f \to M_{\hat{f}} \in \mathcal{B}(L^2(\hat{G}))$, where $M_{\hat{f}}$ denotes the operator given by multiplication $\hat{f}$. Since this is isometric, λ is an isometric isomorphism. Thus $\mathrm{C}^*(G) \cong \mathrm{C}^*_r(G)$. ▮

Note that in this case, if G is discrete, then $C^*_r(G)$ can be given the additional structure of a compact quantum group. This is true more generally (for nonabelian discrete groups), and one of the particularly nice things about quantum groups in general, is that their theory is able to extend this notion of Pontryagin duality beyond the case of abelian groups. Unfortunately, this is beyond the scope of this course, but I think it is a nice motivation for the study of quantum groups.

Crossed products. An important an interesting generalisation of a group C*-algebra is the crossed product of a C*-algebra by a locally compact group G. The construction of the crossed product has a lot of similarities to the construction of group C*-algebras, but now we have to take into account things like the representations of the C*-algebra. Again we will restrict ourselves to unimodular groups.

By an action α of G on a C*-algebra A we always mean a strongly continuous group homomorphism $\alpha : G \to \mathrm{Aut}\,(A)$, where $\mathrm{Aut}\,(A)$ denotes the group of *-automorphisms of A.

6.41 Definition: Suppose that G is a locally compact group acting on a C*-algebra A. A *covariant representation* is a triple (H, π, u) where H is a Hilbert space, (H, π) is a representation for A, (H, u) is a unitary representation for G and π and u satisfy the covariance condition

$$u(g)\pi(a)u(g)^* = \pi(\alpha_g(a)),$$

for every $a \in A$ and $g \in G$.

6.42 For a group action $\alpha : G \to \mathrm{Aut}\,(A)$, the space $L^1(G, A, \alpha)$ is defined as follows.

First, take compactly supported continuous functions $C_c(G, A)$ with twisted convolution

$$(f * g)(t) = \int_G f(s)\alpha_s(g(s^{-1}t)ds.$$

and

$$f^*(s) = (\alpha_s(f(s^{-1})^*)$$

$L^1(G, A, \alpha)$ is the completion with respect to the 1-norm

$$\|f\|_1 = \int_G \|f(s)\| ds.$$

Note that the norm inside the integral is the norm of the C*-algebra A.

6.43 As we did for the group algebras, we can relate a covariant representation for $\alpha : G \to \mathrm{Aut}\,(A)$ to a representation of $L^1(G, A, \alpha)$ on an L^2 space, which can be defined as the completion of $C_c(G, A, \alpha)$ with respect to the norm

$$\|f\|_2 := \left(\int_G \|f(s)\|^2 ds \right)^{1/2}.$$

Given a covariant representation (H, π, u), the *integrated form* of (H, π, u) is defined to be

$$\pi(f)(g) = \int_G \pi(f(s))u(s)g ds$$

where $f \in L^1(G, A, \alpha)$ and $g \in L^2(G, A, \alpha)$.

PROPOSITION: *Let G be a locally compact group, A a C*-algebra and $\alpha : G \to \mathrm{Aut}\,(A)$ an action of G on A. For any covariant representation, the associated integrated form is a representation of $L^1(G, A, \alpha)$ on $H = L^2(G, A, \alpha)$.*

In the opposite direction, a representation $\pi : L^1(G, A)$ also gives a covariant representation of $\alpha : G \to \mathrm{Aut}\,(A)$ using an approximate unit for $L^1(G, A)$. Since the details are similar to the case for unitary representations, they are left as an exercise.

6.44 Given a Hilbert space H_0, the space $L^2(G, H_0)$ consists of square-integrable functions from f to H_0. It is a Hilbert space with respect to the inner product

$$\langle f, g \rangle = \int_G \langle f(s), g(s) \rangle_{H_0} ds.$$

The (left) *regular covariant representation* corresponding to $\pi_0 : A \to H_0$ is (H, π, u) where $H = L^2(G, H_0)$, $\pi : A \to \mathcal{B}(H)$ is given by

$$\pi(a)(f)(s) = \pi_0(\alpha_{s^{-1}})f(s), \qquad (f \in L^2(G, H_0), s \in G);$$

and $u : G \to \mathcal{U}(H)$ is given by

$$u(s)(f)(t) := f(s^{-1}t), \qquad (f \in L^2(G, H_0), t \in G).$$

6.45 DEFINITION: Let $\alpha : G \to \mathrm{Aut}\,(A)$ be an action of locally compact group G on a C*-algebra. Let $\lambda : L^1(G, A, \alpha) \to \mathcal{B}(H)$ denote the direct sum of all integrated forms of regular representations. The *reduced crossed product of A by G*, written $A \rtimes_{r,\alpha} G$, is the closure of $\lambda(L^1(G, A, \alpha)) \subset \mathcal{B}(H)$.

The *full crossed product* of A by G is the closure of $\pi_u(L^1(G, A, \alpha)) \subset \mathcal{B}(H_u)$ where (π_u, H_u) denotes the universal representation, that is, the direct sum of all irreductible representations of $L^1(G, A)$.

6.46 Example: When G acts on a locally compact Hausdorff space X, it induces an action on the C*-algebra $C_0(X)$ by $\alpha_g(f)(x) = f(g^{-1}x)$, $x \in X$, $g \in G$, $f \in C_0(X)$. This provides us with many interesting examples:

(a) Let G act trivially on a point x. Then the associated crossed products $\mathbb{C} \rtimes_r G$ and $\mathbb{C} \rtimes_f G$ are just $C^*_r(G)$ and $C^*_f(G)$, respectively.

(b) Suppose that α is the action of G on itself by translation: $\alpha_g(s) = gs$. Then $C_0(G) \rtimes_{f,\alpha} G = C_0(G) \rtimes_{r,\alpha} G \cong \mathcal{K}(L^2(G))$.

6.47 Theorem: *Let $\alpha : G \to \mathrm{Aut}\,(A)$ be the action of a locally compact abelian group on a unital* C**-algebra A. Then there is an action of $\hat{G}$ on $A \rtimes_{r,\alpha} G$, called the dual action.*

Universal C*-algebras. Interesting examples of C*-algebras are often described as universal objects given by generators and relations. This has to be done with some care, though, because some generators and relations cannot be used to construct C*-algebras. This is because the generators and relations have to be realisable as bounded operators on a Hilbert space. Thus relations, which will usually be algebraic relations among the generators and their adjoints, will require a norm condition.

6.48 Definition: Given a set of generators $\mathcal{G}$ and relations $\mathcal{R}$, a *representation* of $(\mathcal{G}, \mathcal{R})$ on a Hilbert space H is a map $\pi : \mathcal{G} \to \mathcal{B}(H)$ such that $\pi(\mathcal{G})$ are operators satisfying the relations $\mathcal{R}$.

6.49 If $\mathcal{A}$ is the free *-algebra on the relations $\mathcal{R}$ then this induces a representation $(H, \pi_{\mathcal{G}})$ of $\mathcal{A}$ on H. Let $(\mathcal{G}, \mathcal{R})$ be a set of generators and relations and let $\mathcal{A}$ denote the free algebra on generators $\mathcal{G}$. Suppose that, for every $a \in A$,

$$p(a) = \sup\{\|\pi_{\mathcal{G}}(a)\| \mid \pi_{\mathcal{G}} \text{ is a representation of } (\mathcal{G}, \mathcal{R})\}$$

is finite. Then $p(a)$ is a seminorm on on $\mathcal{A}$

6.50 Definition: Given a set of generators and relations $(\mathcal{G}, \mathcal{R})$ such that the $p(a) < \infty$ for every $a \in \mathcal{A}$, where $\mathcal{A}$ and p are defined as above. The *universal* C**-algebra* of $(\mathcal{G}, \mathcal{R})$, written $\mathrm{C}^*(\mathcal{G} \mid \mathcal{R})$, is the enveloping C*-algebra of $(\mathcal{A}, p)$.

6.51 The universal C*-algebra $A = \mathrm{C}^*(\mathcal{G}, \mathcal{R})$ has the following universal property: If a C*-algebra B contains a set of elements X in one-to-one correspondence with $\mathcal{G}$ which also satisifes the relations $\mathcal{R}$, then there is a surjective *-homomorphism $A \to \mathrm{C}^*(X)$ where $\mathrm{C}^*(X)$ is the $\mathrm{C}^*(X)$-subalgebra of B generated by X.

6.52 Examples: 1. The first example is a nonexample: There is no universal C*-algebra generated by a self adjoint element. That is, if $\mathcal{G} = \{a\}$ and $\mathcal{R} = \{a = a^*\}$, then there are no representations of $(\mathcal{G}, \mathcal{R})$ where p defines a seminorm. The reason, of course, is that $p(a)$ will certainly never be finite.

2. Let $\mathcal{G} = \{a, 1\}$ and $\mathcal{R} = \{\|a\| \leq 1, a^* = a, 1 = 1^* = 1^2, 1a = a1 = a\}$. Then $C^*(\mathcal{G} \mid \mathcal{R}) \cong C([-1, 1])$.

3. Let $\mathcal{G} = \{u\}$ and $\mathcal{R} = \{u^*u = uu^* = 1\}$. Then the universal C*-algebra— the universal C*-algebra generated by a unitary—is then isomorphic to $C(\mathbb{T})$. Note how this is related to the the group C*-algebra construction for $\hat{\mathbb{T}} \cong \mathbb{Z}$.

6.53 An important example is the following. Let $n \in \mathbb{N}$ and let $\mathcal{G} = \{e_{i,j} \mid 1 \leq i, j \leq n\} \cup \{1\}$ and $\mathcal{R}$ be the relations $\{e_{ij}e_{kl} = \delta_{jk}e_{il},\, e_{ii}^* = e_{ii}^2 = e^{ii},\, \sum_{i=1}^n e_{ii} = 1\}$ The universal C*-algebra generated by these generators and relations is of course just M_n. Thus, whenever we find elements satisfying these relations (nontrivially) in a given C*-algebra A, we get a copy of M_n sitting inside of A. We call such a set of elements "matrix units".

In this section, we're interested in universal algebras that result in C*-algebras that are, on the one hand, very far removed from the UHF algebras we've already encountered—they are never stably finite, for example—yet on the other hand bear some interesting similarities: they are simple and, like some of our UHF algebras, some of the Cuntz algebras have a certain self-absorbing property.

6.54 Definition: Let A be a unital simple C*-algebra. A is *purely infinite* if for every $x \in A \setminus \{0\}$ there is $a, b \in A$ such that $axb = 1$.

6.55 Definition: Let $n \in \mathbb{N} \setminus \{0\}$ and let $\mathcal{G} = \{s_1, \dots, s_n\}$. Define relations on $\mathcal{G}$ by $\mathcal{R} = \{\sum_{j=1}^n s_j s_j^* = 1, s_i^* s_i = 1, 1 \leq i \leq n\}$. Then the universal C*-algebra on $\mathcal{G}$ subject to $\mathcal{R}$ is well-defined and we call $C^*(\mathcal{G} \mid \mathcal{R})$ the *Cuntz algebra* of type n and denote it by $\mathcal{O}_n$.

6.56 We can also define a Cuntz algebra of type ∞, denoted $\mathcal{O}_\infty$, in the obvious way.

Definition: The Cuntz algebra $\mathcal{O}_\infty$ is the universal C*-algebra generated by a sequence of isometries $(s_i)_{i \in \mathbb{N}}$ such that $\sum_{j=1}^n s_j s_j^* \leq 1$ for every $n \in \mathbb{N}$.

6.57 For $k \in \mathbb{N}$, let W_k^n denote the set of k-tuples $(j_1, \dots, j_k)$ were $j_i \in \{1, \dots, n\}$ if $n < \infty$ and $j_i \in \mathbb{N}$ if $n = \infty$. Let $W_\infty^n = \cup_{k \in \mathbb{N}} W_k^n$.

For $\mu = (j_1, \dots, j_k) \in W_k^n$ we will denote the element $s_{j_1} s_{j_2} \cdots s_{j_k}$ by s_μ. We may also denote 1 by s_0. If $\mu \in W_k^n$ then the length of μ, written $\ell(\mu)$, will be k. If $\mu = 0$ then $\ell(\mu) = 0$.

Lemma: *Let $\mu, \nu \in W_\infty^n$ and let $p = s_\mu s_\mu^*$ and $q = s_\nu s_\nu^*$. Then*

(i) *If $\ell(\mu) = \ell(\nu)$ then $s_\mu^* s_\nu = \delta_{\mu\nu} 1$, and if $s_\mu^* s_\nu \neq 0$ then $s_\mu = s_\nu$ and $p = q$;*

(ii) *if* $\ell(\mu) < \ell(\nu)$ *and* $s_\mu^* s_\nu \neq 0$ *then* $s_\nu = s_\mu s_{\mu'}$ *with* $\mu' \in W^n_{\ell(\nu)-\ell(\mu)}$ *and* $q < p$*;*
(iii) *if* $\ell(\mu) > \ell(\nu)$ *and* $s_\mu^* s_\nu \neq 0$ *then* $s_\mu = s_\nu s_{\nu'}$ *with* $\nu' \in W^n_{\ell(\mu)-\ell(\nu)}$ *and* $p < q$*.*

6.58 Lemma: *If* $w \neq 0$ *is a word in* $\{s_i\} \cup \{s_i^*\}$ *then there are unique elements* $\mu, \nu \in W^n_\infty$ *such that* $w = s_\mu s_\nu^*$*.*

6.59 Let $F_0^n = \mathbb{C}$ and for $k > 0$ let $F_k^n = C^*(s_\mu s_\nu^* \mid \mu, \nu \in W_k^n)$. Let $F^n = \cup_{k \in \mathbb{N}} F_k^n$

Proposition: *If* $n < \infty$ *then* $F_k^n \cong M_{n^k}$*. Moreover* $F_k^n \subset F_{k+1}^n$ *so* $F^n \cong \mathcal{U}_{n^\infty}$*, the UHF algebra of type* n^∞*. If* $n = \infty$ *the* $F_k^n \cong \mathcal{K}$ *and* F^n *is an AF algebra.*

6.60 Let P be the *-algebra generated by $\{s_1, \dots, s_n\} \cup \{s_1^*, \dots, s_n^*\}$. If $w = s_\mu s_\nu^*$ is a word in $\{s_i\} \cup \{s_i^*\}$ then

(i) if $\ell(\mu) - \ell(\nu) > 0$ then $w(s_1^*)^k = s_\nu s_\mu (s_1^*)^k \in F^n_{\ell(\mu)}$ and thus $w \in F^n_{\ell(\mu)} s_1^k$;
(ii) if $\ell(\mu) - \ell(\nu) < 0$ then $(s_1^*)^k w = (s_1^*)^k s_\mu s_\nu^* \in F^n_{\ell(\nu)}$ and thus $w \in s_1^k F^n_{\ell(\nu)}$;
(iii) if $\ell(\mu) = \ell(\nu)$ then $w \in F_r^n = F_s^n$. Thus, since any $a \in P$ is a linear combination of words, we can write

$$a = \sum_{i=-N}^{-1} s_1^i a_i + a_0 + \sum_{i=1}^{n} a_i s_1^i$$

where the $a_i \in F^n$. Set $F_i(a) = a_i$.

6.61 Let A be a C*-algebra and $B \subset A$ a C*-subalgebra. A *conditional expectation* is a c.p.c map $\Phi : A \to B$ satisfying $\Phi(b) = b$ for every $b \in B$ and $\Phi(b_1 a b_2) = b_1 \Phi(a) b_2$ for every $b_1, b_2 \in B$ and $a \in A$. (This last requirement is equivalent to saying that Φ is a left- and right- B module map.) A conditional expectation Φ is faithful when $\Phi(a^* a) = 0$ if and only if $a = 0$. (A google search for faithful conditional expectation leads to some amusing results.)

6.62 Proposition: *The map* $F_0 : P \to F^n$ *is a faithful conditional expectation.*

6.63 Proposition: *Suppose that* $n < \infty$ *and* $x \in \mathcal{O}_n$ *is nonzero. Then there exist* $a, b \in \mathcal{O}_n$ *such that* $axb = 1$*.*

6.64 Proposition: $\mathcal{O}_n$ *is simple and purely infinite.*

Exercises

6.1 Characterise all finite-dimensional C*-algebras.

6.2 Let $A = \varinjlim(A_n, \phi_n)$ be the inductive limit of a sequence of C*-algebras and let B be a C*-algebra. For every $n \in \mathbb{N}$ let $\psi^{(n)} : A_n \to B$ be a *-homomorphism satisfying $\psi^{(n+1)} \circ \phi_n = \psi^{(n)}$ and let $\psi : A \to B$ be the induced *-homomorphism. Then

(i) ψ is injective if and only if $\ker(\psi^{(n)}) \subset \ker(\phi^{(n)})$ for every $n \in \mathbb{N}$, and

(ii) ψ is surjective if and only if $B = \overline{\cup_{j=1}^{\infty} \psi^{(n)}(A_n)}$.

6.3 Let (A_n, ϕ_n) be an inductive sequence of C*-algebras. Let $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ be an increasing sequence of natural numbers. Show that

$$\varinjlim(A_n, \phi_n) \cong \varinjlim(A_{n_k}, \phi_{n_k,n_{k+1}}).$$

6.4 Show that the definition of a UHF algebra $\mathfrak{p}$ is independent of the choice of UHF sequence $(n_k)_{k\in\mathbb{N}}$ for $\mathfrak{p}$. Thus any UHF algebra is uniquely identified with a supernatural number $\mathfrak{p}$

6.5 Show that a UHF algebra is nuclear (see exercise 5.4).

6.6 Let $\mathcal{U}_\mathfrak{q}$ be a UHF algebra of infinite type $\mathfrak{p}$. Show that $\mathcal{U}_\mathfrak{p} \otimes \mathcal{U}_\mathfrak{p} \cong \mathfrak{U}_\mathfrak{p}$. Suppose that $\mathfrak{p}$ divides $\mathfrak{q}$. Show that $\mathcal{U}_\mathfrak{p}$ absorbs $\mathfrak{U}_\mathfrak{q}$ in the sense that $\mathcal{U}_\mathfrak{q} \otimes \mathcal{U}_\mathfrak{p} \cong \mathcal{U}_\mathfrak{q}$. The *universal* UHF algebra, denoted $\mathfrak{Q}$ is the UHF algebra associated to the supernatural number $\Pi_{p \text{ prime}}\, p^\infty$. Thus the universal UHF algebra absorbs all other UHF algebras (including all matrix algebras!).

6.7 Let $\mathfrak{p} = 2^\infty$. The associated C*-algebra is often called the CAR algebra (where CAR stands for *canonical anticommutation relations.* Let X be a compact Hausdorff space. Show that the C*-algebra of functions on X taking values in $\mathcal{U}_{2^\infty}$, that is, $C(X, \mathcal{U}_{2^\infty})$, absorbs $\mathcal{U}_{2^\infty}$.

6.8 A unital C*-algebra A has *real rank zero* if the invertible self-adjoint elements are dense in A_{sa}. There is also a notion of real rank n for $n \in \mathbb{N}$. Real rank is a generalisation of covering dimension to C*-algebras.

(a) Let X be the Cantor set. Show that X has covering dimension 0. Show that $[0, 1]$ has covering dimension 1. Show that $C(X)$ has real rank zero but $C([0, 1])$ does not. (See exercise 4.5.)

(b) Let $a \in M_n = M_{n\times n}(\mathbb{B})$, $b \in M_{n\times 1}(\mathbb{C})$, $c \in M_{1\times n}(\mathbb{C})$ and $d \in \mathbb{C}$. Let $\epsilon > 0$. Suppose that d is invertible and there is $a' \in M_n$ that is invertible which satisfies $\|a' - (a - bd^{-1}c)\| < \epsilon$. Show that

$$\begin{pmatrix} a' + bd^{-1}c & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} I_n & 0 \\ -d^{-1}c & 1 \end{pmatrix},$$

(as elements in M_{n+1}) and

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a' + bd^{-1}c & b \\ c & d \end{pmatrix} \right\| < \epsilon.$$

(c) Let A be a unital C*-algebra. Suppose $b \in \mathrm{Inv}(A)$ and there is $a \in A_{sa}$ with $\|a - b\| < \epsilon$. Show that there is $b' \in A_{sa} \cap \mathrm{Inv}(A)$ with $\|a - b'\| < \epsilon$.

(d) Prove that any UHF algebra has real rank zero.

6.9 We saw that all UHF algebras are simple. Is the same true for AF algebras? Give a proof or counterexample.

7. A very short introduction to classification for simple nuclear C*-algebras

In this section we will try to sketch the proof of Elliott's classification of unital AF algebras by ordered K-theroy and then given an idea of some more recent research directions in the classification and structure of C*-algebras.

Let A be a unital C*-algebra and let $M_\infty(A) = \cup_{n\in\mathbb{N}} M_n(A)$ where $M_n(A)$ is included into $M_{n+1}(A)$ by copying M_n into the top left corner of $M_{n+1}(A)$, that is

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

If $p \in M_n(A)$ and $q \in M_m(A)$ the we define $p \oplus q \in M_{n+m}(A)$ to be

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

For any $p, q \in M_\infty(A)$, we extend the Murray–von Neumann equivalence by setting $p \sim q$ if there is some $m, n \in \mathbb{N}$ and some $v \in M_{m,n}$ such that $v^*v = p$ and $vv* = q$. In the stably finite case, $p \sim q$ if and only if there is some $k \in \mathbb{N}$ such that $p \oplus 1_k \sim q \oplus 1_k$.

Let $[p]$ denote the equivalence class of the projection $p \in M_{\infty(A)}$ and let $\mathcal{P}(A)$ be all Murray–von Neumann equivalence classes of projections in $M_\infty(A)$. Since $p\oplus q$ is equivalent to $q \oplus p$, $\mathcal{P}(A)$ has the structure of an abelian semigroup. $K_0(A)$ is defined to be the Grothendieck group of $\mathcal{P}(A)$:

$$K_0(A) = \{[p] - [q] \mid p, q \in M_\infty(A)\}.$$

Note that, by the Grothendieck construction, $[p] - [q] = [p'] - [q']$ in $K_0(A)$ if and only if there is $r \in \mathcal{P}(A)$ such that $[p] + [q'] + [r] = [p'] + [q] + [r]$.

7.1 Exercise: Suppose that $p, q \in M_n(A)$ are orthogonal projections, that is, $pq = qp = 0$. Then $p + q$ is a projection in $M_n(A)$. Show that $[p] + [q] \sim [p + q]$.

7.2 An *ordered abelian group* is an abelian group G together with a partial order that respects the group structure in the sense that if $x \leq y$ then $x + z \leq y + z$ for every $x, y, z \in G$ and such that

$$G = \{x \in G \mid 0 \leq x\} - \{x \in G \mid 0 \leq x\}.$$

As in the case for C*-algebras, we will denote the positive elements of G by G_+.

7.3 A *cone* in an abelian group G is a subset H such that $H+H \subset H$, $G = H - H$ and $H \cap (-H) = \{0\}$. If G has a partial order, then G_+ is a cone. Conversely, if H is a cone in G, then setting $x \leq y$ if and only if $y - x \in H$, defines a partial order on G.

7.4 THEOREM: *Let A be a unital stably finite* C*-*algebra. Then*

$$K_0(A)_+ := \{[p] \mid p \in \mathcal{P}(A)\}$$

is a cone and hence $K_0(A)$ is an ordered abelian group.

PROOF: That $K_0(A)_+ + K_0(A)_+ \subset K_0(A)_+$ and $K_0(A) = K_0(A)_+ - K_0(A)_+$ is immediate. So we just need to show that $K_0(A)_+ \cap (-K_0(A)_+) = \{0\}$. If $p \oplus q \in M_n(A)$, then $[1_n] + [p \oplus q] = [1_n]$ and so $[1_n - p \oplus q] = 1_n$. There is $m \in \mathbb{N}$ such that $(1_n - p \oplus q) \oplus 1_m$ is Murray–von Neumann equivalent to $1_n \oplus 1_m$, that is, there exists $v \in M_{n+m}(A)$ with $v^*v = (1_n - p \oplus q) \oplus 1_m$ and $vv^* = 1_{n+m}$. Since A is stably finite, we must have $v^*v = 1_{n+m}$. Thus $m = 0$ and $p \oplus q = 0$, which means $p = 0$ and $q = 0$. So $[p] = [q] = 0$, which proves the theorem. ▌

7.5 An *order unit* for an ordered abelian group G is an element $u \in G_+$ such that, for every $x \in G$ there exists an $n \in \mathbb{N}$ such that $-nu \leq x \leq nu$.

PROPOSITION: *Let A be a unital stably finite* C*-*algebra. Then $[1_A]$ is an order unit for $K_0(A)$.*

PROOF: Exercise. ▌

7.6 COROLLARY: *Let A be a unital AF algebra. Then $(K_0(A), K_0(A)_+, [1_A])$ is an ordered abelian group with distinguished order unit.*

7.7 If (G, G_+, u) and (H, H_+, v) are ordered abelian groups with distinguished order units, then a unital positive homomorphism is a homomorphism $\phi : G \to H$ satisfying $\phi(G_+) \subset H_+$ and $\phi(u) = v$. If ϕ is a group isomorphism and ϕ^{-1} is also a unital positive homomorphism, then we call ϕ a *unital order isomorphism.*

Let A abd B be C*-algebras and $\phi : A \to B$ a *-homomorphism. Define

$$\phi_* : K_0(A) \to K_0(B)$$

by setting $\phi([p] - [q]) = [\phi(p)] - [\phi(q)]$. (It is easy to see that this is well-defined.) We also have that $\phi(K_0(A)_+) \subset K_0(B)_+$, and if ϕ is unital, then $\phi_*([1_A]) = [\phi(1_A)] = [1_B]$ so ϕ_* preservers the order unit. If $\phi : A \to B$ is an isomorphsim, then ϕ_* is a unital order isomorphism.

7.8 THEOREM: *Let $F = M_{n_1} \oplus \cdots \oplus M_{n_m}$ be a finite dimensional* C*-*algebra and let $e_{ij}^{(k)}$, $1 \leq i, j \leq n_k$, $1 \leq k \leq K$, be matrix units for F. The map*

$$\phi : (\mathbb{Z}^m, \mathbb{Z}^m_+, (n_1, \ldots, n_m)) \to (K_0(A), K_0(A)_+, [1_A])$$

given by

$$(r_1, \dots, r_m) \to \sum_{k=1}^{m} r_k[e_{11}^{(k)}]$$

is a unital order isomorphism.

PROOF: If $p \in \mathcal{P}(F)$ then there are $p_k \in \mathcal{P}(M_{n_k})$, $1 \leq k \leq m$ such that $p = \sum_{k=1}^m p_k$. Each p_k is a projection in a matrix algebra over M_{n_k} where the equivalence classes of projections are determined by their ranks. Thus $p_k \sim r_k[e_{11}^{(k)}]$ for some $r_k \geq 0$ and so $p \sim \sum_{k+1}^m r_k[e_{11}^{(k)}]$. It follows that the map ϕ is surjective and that ϕ is positive. It is also clear that $\phi((n_1, \dots, n_m)) = \sum_{k=1}^m n_k[e_{11}^{(k)}] = \sum_{k=1}^m [1_k] = [1]$.

Let $\pi_k : F \to M_{n_k}$ be the surjection onto M_{n_k}. Suppose $\phi((r_1, \dots, r_m)) = 0$. Then $\sum_{k=1}^m r_k[e_{11}^{(k)}] = 0$ so for each $l \in \{1, \dots, m\}$ $0 = (\pi_l)_*(\sum_{k=1}^m r_k[e_{11}^{(k)}]) = r_l[e_{11}^{(l)}]$. Thus the direct sum of r_k copies of $e_{11}^{(l)}$ is equivalent to zero. This is only possible if $r_l = 0$. So ϕ is injective. ∎

7.9 DEFINITION: A C*-algebra A has the *cancellation property* if whenever $p, q \in \mathcal{P}(A)$ and $[p] = [q]$ in $K_0(A)$, then $p \sim q$, or, equivalently whenever $[p] + [r] = [q] + [r]$ then $[p] = [q]$.

7.10 PROPOSITION: *Every AF algebra has the cancellation property.*

7.11 LEMMA: *Let A be* C**-algebra with the cancellation property. $q \in A$ be a projection and $p_1, \dots, p_n \in \mathcal{P}(A)$ such that $[q] = [p_1 \oplus \cdots \oplus p_n]$. Then there are are pairwise orthogonal projections $p_1', \dots, p_n' \in A$ such that q is Murray–von Neumann equivalent to $\sum_{i=1}^n p_i'$ and each p_i' is Murray–von Neumann equivalent to p_i.*

PROOF: It suffices to prove that if $r \in M_n(A)$, $p \in A$ and $q \in A$ satisfy $[r]+[p] = [q]$ then there is a $p' \in A$ such that $p'p = pp' = 0$ and $[r] = [p']$. Let $s \in M_m(A)$ satisfy $[s] = [q] - [p] - [r]$. Since A has the cancellation property, $s \oplus r \sim q - p$ so there is $v \in M_{1,m+n}$ such that $v^*v = r \oplus s$ and $vv^* = q - p$. Let $p' = v(r \oplus 0_m)v^*$. Then $p' \in A$ and with $w = (r \oplus 0_m)^{1/2}v^*$ we have $w^*w = p'$ and $w^*w = r \oplus 0_m$, so $[p'] = [r]$. Since $p' = v(r \oplus 0_m)v^* \leq v(1_{n+m})v = vv^* = q - p$, hence also $p' \leq q$ thus $p'p = pp' = 0$, as required. ∎

7.12 THEOREM: *Let $A = M_{n_1} \oplus \cdots \oplus M_{n_m}$ and B be another finite dimensional* C**-algebra. Suppose that $\phi : (K_0(A), K_0(A)_+, [1_A]) \to (K_0(B), K_0(B)_+, [1_B])$ is a unital positive homomorphism. Then there exists a homomorphism $\Phi : A \to B$ such $\Phi_* = \phi$. Moreover, Φ is unique up to conjugation by a unitary.*

PROOF: Let $e_{ij}^{(l)}$ be a set of matrix units for $M_{n_l} \subset M_{n_1} \oplus \cdots \oplus M_{n_m}$, $0 \leq l \leq m$. Denote by 1_l the unit of M_{n_l}. Since ϕ is positive, $\phi([1_l]) = [p_l]$ for some projection

$p_l \in \mathcal{P}(A)$. Thus

$$[p_1 \oplus \cdots \oplus p_m] = \phi(\sum_{l=1}^{m}[1_l]) = \phi([1_a]) = [1_B].$$

Since B has the cancellation property, there are m mutually orthogonal projections $q_1, \dots, q_m \in B$ such that $\sum_{l=1}^m q_l = [1_B]$ and $[\phi(1_l)] = [p_l] = [q_l]$ for every $1 \leq l \leq m$.

Similarly, we have that $\phi([e_{11}^{(l)}]) = [p_{11}^{(l)}$ for some $p_{11}^{(l)} \in \mathcal{P}(A)$. Thus $[q_l] = \phi([1_l]) = \phi(n_l[e_{11}^{(l)}]) = [p_{11}^{(l)} \oplus \cdots \oplus p_{11}^{(l)}]$. Since $q_l \in B$, there are mutually orthogonal $q_{11}^{(l)}, \dots, q_{n_l,n_l}^{(l)} \in B$ with $[q_{ii}^{(l)}] = [p_1 1^{(l)}] = \phi([e_{11}^{(l)}])$. Since $q_{ii}^{(l)} \sim q_{11}^{(l)}$ for each $1 \leq i \leq n_l$, there are $v_i \in B$ with $v_i^{(l)}(v_i^{(l)})^* = q_{ii}^{(l)}$ and $(v_i^{(l)})^* v_i^{(l)} = q_{11}^{(l)}$. Put

$$q_{ij}^{(l)} = v_i^{(l)}(v_j^{(l)})^*.$$

One can verify that, for each l, $q_{ij}^{(l)}$ satisfy the matrix relations for M_{n_l}. Since the q_l are pairwise orthogonal, this gives a map from the generators of A to B and hence a map $\Phi : A \to B$. Moreover, we get that $\Phi_*([e_{11}^{(l)}]) = [\Phi(e_{11}^{(l)}] = [q_{11}^{(l)}] = \phi([e_{11}^{(l)}]$. Since $[e_{11}^{(l)}]$ $1 \leq l \leq m$ generate $K_0(A)$, this implies $\Phi_* = \phi$.

Suppose now that $\Phi, \Psi : A \to B$ are both unital homomorphisms satisfying $\Phi_* = \Psi_*$. Let $p_{ij}^{(l)} = \Phi(e_{ij}^{(l)}$ and $q_{ij}^{(l)} = \Psi(e_{ij}^{(l)})$, $1 \leq i, j, \leq n_l$, $1 \leq l \leq m$. Then $[p_{ij}^{(l)}] = \Phi_*([e_{ij}^{(l)}]) = \Psi_*([e_{ij}^{(l)}]) = [q_{ij}^{(l)}]$, so $p_{ij}^{(l)} \sim q_{ij}^{(l)}$. Thus there are $v_l \in B$, $1 \leq l \leq m$ satisfying $v_l^* v_l = p_{11}^{(l)}$ and $v_l v_l^* = q_{11}^{(l)}$. Set

$$w = \textstyle\sum_{l=1^m} \sum_{i=1}^{n_l} q_{il}^{(l)} w_l p_{li}^{(l)}.$$

Then w is a unitary and $wp_{ij}^{(l)}w^* = q_{ij}^{(l)}$ for every $1 \leq i, j \leq n_l$ and $1 \leq l \leq m$. Thus $\Psi * (e_{ij}^l) = w\Phi(e_{ij}^{(l)})w^*$ for every $1 \leq i, j \leq n_l$ and $1 \leq l \leq m$. Since these elements generate A, we have $\Psi = \mathrm{ad}\,(w) \circ \Phi$. ▮

7.13 Lemma: *Let A, B and C be unital stably finite* C*-*algebras, with A finite-dimensional. If $\phi : K_0(A) \to K_0(C)$ and $\psi : K_0(B) \to K_0(C)$ are positive homomorphism which satisfy $\phi(K_0(A)_+) \subset \psi(K_0(A)_+)$. The there is a positive homomorphism $\rho : K_0(A) \to K_0(B)$ such that $\psi \circ \rho = \phi$.*

Proof: Exercise. ▮

7.14 Lemma: *Suppose that $A = \varinjlim(A_n, \phi_n)$ and $B = \varinjlim(B_n, \psi_n)$ with each ϕ_n, ψ_n injective and there are* *-*homomorphisms $\alpha_n : A_n \to B_n$ and $\beta_n : B_n \to A_{n+1}$*

making the following diagram commute:

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{\phi_1} & A_2 & \xrightarrow{\phi_2} & A_3 & \longrightarrow & \cdots & \longrightarrow & A \\
{\scriptstyle\alpha_1}\downarrow & \overset{\beta_1}{\nearrow} & {\scriptstyle\alpha_2}\downarrow & \overset{\beta_2}{\nearrow} & {\scriptstyle\alpha_3}\downarrow & \overset{\beta_2}{\nearrow} & & & \\
B_1 & \xrightarrow{\psi_1} & B_2 & \xrightarrow{\psi_2} & B_3 & \longrightarrow & \cdots & \longrightarrow & B.
\end{array}$$

Then there are *-isomorphisms $\alpha : A \to B$ and $\beta : B \to A$ making

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{\phi_1} & A_2 & \xrightarrow{\phi_2} & A_3 & \longrightarrow & \cdots & \longrightarrow & A \\
{\scriptstyle\alpha_1}\downarrow & \overset{\beta_1}{\nearrow} & {\scriptstyle\alpha_2}\downarrow & \overset{\beta_2}{\nearrow} & {\scriptstyle\alpha_3}\downarrow & \overset{\beta_2}{\nearrow} & & & {\scriptstyle\beta}\uparrow\ \downarrow{\scriptstyle\alpha} \\
B_1 & \xrightarrow{\psi_1} & B_2 & \xrightarrow{\psi_2} & B_3 & \longrightarrow & \cdots & \longrightarrow & B.
\end{array}$$

commute.

7.15 Lemma: *Let $A = \varinjlim(A_n, \phi_n)$ be an AF algebra and let F be a finite dimensional algebra. Suppose that there are positive homomorphisms $\alpha : K_0(A_1) \to K_0(F)$ and $\gamma : K_0(F) \to K_0(A)$ such that $\gamma \circ \alpha = \phi_*^{(1)}$. Then there is $n \in \mathbb{N}$ and a positive group homomorphism $\beta : K_0(F) \to K_0(A_n)$ such that*

$$\begin{array}{ccccc}
K_0(A_1) & \xrightarrow{(\phi_{1,n})_*} & K_0(A_n) & \xrightarrow{\phi_*^{(n)}} & K_0(A) \\
& {\scriptstyle\alpha_1}\searrow & {\scriptstyle\beta}\uparrow & \nearrow{\scriptstyle\gamma} & \\
& & K_0(F) & &
\end{array}$$

commutes. Moreover, if the maps ϕ_n are unital and $\alpha([1_A]) = [1_F]$, then also $\beta([1_F]) = [1_{A_n}]$.

7.16 Theorem: [Elliott] *Let A and B be unital approximately finite C*-algebras. Any *-isomorphism $\Phi : A \to B$ induces an order isomorphism of K_0-groups, $\Phi_* : (K_0(A), K_0(A)_+, [1_A]) \to (K_0(B), K_0(B)_+, [1_B])$.*

Conversely, if $\phi : (K_0(A), K_0(A)_+, [1_A]) \to (K_0(B), K_0(B)_+, [1_B])$ is an order isomorphism, then there is a *-isomorphism $\Phi : A \to B$ satisfying $\Phi_* = \phi$.

Proof: Let $(A_n, \psi_n)_{n\in\mathbb{N}}$ and $(B_n, \rho_n)_{n\in\mathbb{N}}$ be inductive limit sequences of finite-dimensional C*-algebras with limits A and B respectively. We may assume that the maps ψ_n and ρ_n are unital and injective.

Consider the finite dimensional C*-algebra A_1. Since $\psi^{(1)}$ is a unital homomorphism, it induces a unital positive map $\psi_*^{(1)} : K_0(A_1) \to K_0(A)$ and thus, by composition with ϕ we have $\phi \circ \psi_*^{(1)} : K_0(A_1) \to K_0(B)$. We have that $\phi \circ \psi_*^{(1)}(K_0(A)_+) \subset K_0(B)_+$ so for large enough n_1, in fact $\phi \circ \psi_*^{(1)}(K_0(A)_+) \subset \rho_*^{(n_1)} K_0(B_{n_1})_+$. Thus, by Lemma 7.13 there is

$$\alpha_1 : K_0(A_1) \to K_0(B_{n_1})$$

satisfying $\rho_*^{(n_1)} \circ \alpha_1 = \phi \circ \psi_*^1$, hence $\phi^{-1} \circ \rho_*^{(n_1)} \circ \alpha_1 = \psi_*^1$ and we may apply Lemma 7.15 to find a $m_1 \in \mathbb{N}$ and a map $\beta_1 : K_0(B_{n_1}) \to K_0(A_{m_1})$ with $\psi^{(m_1)} \circ \beta_1 = \phi^{-1} \circ \rho_*^{(n_1)}$. Thus $\phi \circ \psi^{(m_1)} \circ \beta_1 = \rho_*^{(n_1)}$ so by applying Lemma 7.15 again, we have $n_2 > n_1$ and a map $\alpha_2 : A_{m_1} \to B_{n_1}$ such that $\phi \circ \psi^{(m_1)} = \rho_*^{(n_1)} \circ \alpha_2$.

Continuing the same way, we find $n_1, n_2, n_3, \dots$ and $m_1, m_2, m_3, \dots$ giving the following commutative diagram

$$\begin{array}{ccccccccc} K_0(A_1) & \longrightarrow & K_0(A_{m_1}) & \longrightarrow & K_0(A_{m_2}) & \longrightarrow & \cdots & \longrightarrow & K_0(A) \\ \downarrow{\scriptstyle \alpha_1} & \overset{\beta_1}{\nearrow} & \downarrow{\scriptstyle \alpha_2} & \overset{\beta_2}{\nearrow} & \downarrow{\scriptstyle \alpha_3} & \overset{\beta_3}{\nearrow} & & & {\scriptstyle \phi^{-1}}\uparrow \; \downarrow{\scriptstyle \phi} \\ K_0(B_{n_1}) & \longrightarrow & K_0(B_{n_2}) & \longrightarrow & K_0(B_{n_3}) & \longrightarrow & \cdots & \longrightarrow & K_0(B) \end{array}$$

By Exercise 7.16, the subsequences $(A_{m_1}, \psi_{m_1,m_2})$ and $(B_{n_1}, \psi_{n_1,n_2})$ have inductive limit A and B, respectively. Thus we will relabel A_{m_k} by A_k and B_{n_k} by B_k as well as the connecting maps accordingly.

By Theorem there is a *-homomorphism $\sigma_1 : A_1 \to B_1$ such that $(\sigma)_* = \alpha$ and $\tau : B_1 \to A_2$ with $\tau_* = \beta$ By commutativity of the diagram above, we have that $(\tau_1 \circ \sigma_1)_* = (\psi_1)_*$ so there is a unitary $u_2 \in A_2$ such that $\mathrm{ad}\,(u)\circ$ ∎

7.17 What about arbitrary unital C*-algebras? Can they be classified by K_0? The answer is no, as soon as one moves to more complicated C*-algebras, K_0 (even as a unital ordered group) is not enough to distinguish two C*-algebras. For example, suppose that $A_n = C(\mathbb{T}, F_n)$ where F_n is a finite dimensional C*-algebra and suppose we have *-homomorphisms $\phi_n : A_n \to A_{n+1}$. The inductive limit $A = \lim(A_n, \phi_n)$ is called an A$\mathbb{T}$ algebra ("approximately circle" algebra). In the simple unital case, to distinguish two AT algebras, we need to include the K_1-group and tracial state space.

If A is a unital C*-algebra, then $K_1(A) = K_0(SA)$ where $SA = \{f : [0,1] \to A \mid f(0) = f(1) = 0\}$ is called the *suspension* of A. The K_1 group can also be described by equivalence classes of unitaries in $M_n(A)$ in a similar manner to the way K_0 is defined for projections. AF algebras always have trivial K_1, so this is why it does not play a role in the classification invariant for AF algebras.

7.18 How far, then, can we get if we throw K_1-groups and tracial states into the mix? In fact, quite far! But we'll need a few definitions first.

First of all, it makes sense to look at simple C*-algebras. We should be able to classify these before we can say anything in greater generality. Let's also stick to the separable case. If things are nonseparable, its unlikely that any invariant will be in any sense computable. As we've often seen so far, it is usually easier to deal with unital C*-algebras. The final thing we need is some sort of "finiteness" condition. The first guess for such a condition was to restrict to those C*-algebras with the completely positive approximation property.

DEFINITION: A C*-algebra A has the *completely positive approximation property* if, for every finite subset $\mathcal{F} \subset A$ and every $\epsilon > 0$, there is a finite-dimensional C*-algebra F and completely positive contract maps $\psi : A \to F$ and $\phi : F \to A$ such that

$$\|\phi \circ \psi(a) - a\| < \epsilon \text{ for every } a \in \mathcal{F}.$$

8. EXTRA MATERIAL

Multiplier algebras. 8.1 A left multiplier L of A is a bounded linear operator $L : A \to A$ which satisfies $L(ab) = L(a)b$ for every $a, b \in A$. Similarly one defines a right multiplier $R : A \to A$ as a bounded linear operator satisfying $R(ab) = aR(b)$ for every $a, b \in A$. To define the multiplier algebra of A, we consider pairs of left and right multipliers (L, R) with the compatibility condition $aL(b) = R(a)b$ for every $a \in A$. The pair (L, R) is called a *double centraliser*. We will denote the set of such pairs by $M(A)$ and show that this is a unital C*-algebra containing A as an *essential* ideal. An ideal is called essential if it has nonempty intersection with every other ideal in A. .

8.2 PROPOSITION: *Let $(L, R) \in M(A)$. Then $\|L\| = \|R\|$, so we define*

$$\|(L, R)\| := \|L\|.$$

PROOF: Note that $\|L(b)\| = \sup_{a,\|a\|\leq 1} \|aL(b)\|$. From this we have

$$\begin{aligned} \|L(b)\| &= \sup_{a,\|a\|\leq 1} \|aL(b)\| \\ &= \sup_{a,\|a\|\leq 1} \|R(a)b\| \\ &\leq \sup_{a,\|a\|\leq 1} \|R(a)\|\|b\| \\ &= \|R\|\|b\| \end{aligned}$$

Thus $\|L\| = \sup_{b,\|b\|\leq 1} \|L(b)\| \leq \|R\|$. One shows similarly that $\|R\| \leq \|L\|$ from which the result follows. ∎

It is easy to check that we can give $M(A)$ the structure of a vector space by viewing it as a closed subspace as $B(A) \oplus B(A)$. To show that $M(A)$ is a C*-algebra we need to define the multiplication and adjoint and then check that the norm above is indeed a C*-norm.

8.3 Let $L : A \to A$ be a bounded operator. Define $L^* : A \to A$ by

$$L^*(a) = L(a^*)^*.$$

The adjoint of $(L,R) \in M(A)$ is then just $(L,R)^* = (R^*, L^*)$. For $(L_1,R_1),(L_2,R_2) \in M(A)$, let

$$(L_1,L_2)(R_1,R_2) = (L_1L_2, R_2, R_1).$$

It is not hard to check that, with these operations and the norm defined above $M(A)$ is a unital C^*-algebra with unit $(\mathrm{id}, \mathrm{id})$. We also have the following.

8.4 Given any $a \in A$, we can define $(L_a, R_a) \in M(A)$ by $L_a(b) = ab$ and $R_a(b) = ba$ for $b \in A$. PROPOSITION: *Let A be a* C*-*algebra . Then*

$$A \to M(A) : a \to (L_a, R_a)$$

identifies A an essential ideal inside $M(A)$, which is proper if and only if A is nonunital.

8.5 There are many important uses for the multiplier algebra, but often it is a little too big and unwieldy. For example, if X is a locally compact Hausdorff space which is not compact, then $M(C_0(X)) \cong C(\beta X)$ where βX is the Stone–Čech compactification of X.

More about representations. 8.6 Let (H,π) be a representation of a C*-algebra A. A vector $\xi \in H$ is called *cyclic* if the linear span of $\{\pi(a)\xi \in H \mid a \in A\}$, which we will denote by $\pi(A)\xi$ (one might think of this as the orbit of ξ under $\pi(A)$) is dense in H. If such a ξ exists, then (H,π) is called a cyclic representation.

THEOREM: *Let $\phi : A \to \mathbb{C}$ be a positive linear functional. The GNS representation associated to ϕ is cyclic with cyclic vector ξ_ϕ, where ξ_ϕ is the unique vector satisfying*

$$\phi(a) = \langle \pi_\phi(a), \xi_\phi \rangle_\phi$$

for every $a \in A$.

8.7 Two representations (H_1,π_1) and (H_2,π_2) of a C*-algebra A are *unitarily equivalent* if there is a unitary operator $u : H_1 \to H_2$ such that $\pi_2(a) = u\pi_1(a)u^*$ for every $a \in A$. As the name suggests, this is an equivalence relation on the representations of A.

8.8 PROPOSITION: *Let (H_1,π_1) be a cyclic representation of A with cyclic vector ξ and let (H_2,π_2) be a cyclic representation of A with cyclic vector μ. Then the following are equivalent:*

(i) *(H_1,π_1) and (H_2,π_2) are unitarily equivalent with unitary $u : H_1 \to H_2$ satisfying $u\xi = \mu$,*
(ii) *$\langle \pi_1(a)\xi, \xi\rangle = \langle \pi_2(a)\mu, \mu\rangle$ for every $a \in A$.*

PROOF: That (i) $\implies$ (ii) is clear. Conversely, suppose that $\langle \pi_1(a)\xi, \xi\rangle = \langle \pi_2(a)\mu, \mu\rangle$ for every $a \in A$. ∎

8.9 A representation (H, π) of A is called *nondegenerate* if the linear span of $\{\pi(a)\xi \mid a \in A, \xi \in H\}$, denoted by $\pi(A)(H)$, is dense in H, or, equivalently, for each $\xi \in H \setminus \{0\}$ there is $a \in A$ such that $\pi(a)(\xi) \neq 0$. Clearly a cyclic representation is nondegenerate, but a nondegenerate representation need not be cyclic. However, we shall see that it is always the direct sum of cyclic representations.

8.10 THEOREM: *Let A be a C^*-algebra. Any nondegenerate representation (H, π) is the direct sum of cyclic representations.*

PROOF: Let $\xi \in H$ be nonzero and let $H_\xi = \overline{\pi(A)\xi}$. By the Kuratowski–Zorn lemma, there is a maximal set $S \subset H \setminus \{0\}$ such that H_ξ, $\xi \in S$ are pairwise orthogonal. Each H_ξ is cyclic and $\pi(A)$-invariant. Their direct sum is a representation on $\oplus_{\xi \in S} H_\xi \cong \cup_{\xi \in S} H_\xi \subset H$. We will show this is all of H. Let $\eta \in (\cup_{\xi \in S} H_\xi)^\perp$. Then it is easy to check that H_η will be orthogonal to each of H_ξ. We have $\pi(a)(\eta) \in H_\eta$ for every $a \in A$ and hence by maximality of S, we must have that $\pi(a)(\eta) = 0$ for every $a \in A$. Since (H, π) is nondegenerate this means $\eta = 0$. Hence $\cup_{\xi \in S} H_\xi = H$. ∎

8.11 A representation $\pi : A \to \mathcal{B}(H)$ is *irreducible* if $K \subset H$ closed vector subspace with $\pi(a)K \subset K$, then $K \in \{0, H\}$.

THEOREM:

Tensor products for C*-algebras. 8.12 Defining a norm on the tensor product is a tricky matter since in general there can be more than one. Here, we construct the *minimal tensor product*, or *spatial tensor product*.

We begin with the tensor product of two Hilbert spaces. Let H and K be Hilbert spaces and form the vector space tensor product $H \otimes K$.

PROPOSITION: *Let H and K be Hilbert spaces. Then there is a unique inner product on $H \otimes K$ such that*

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle = \langle \xi, \xi' \rangle_H \langle \eta, \eta' \rangle_K,$$

for every $\xi, \xi' \in H$ and $\eta, \eta' \in K$.

PROOF:

∎

8.13 The inner product above makes $H \otimes K$ into a pre-Hilbert space, and so we complete it to a Hilbert space, denoted $H \hat{\otimes} K$. The norm on $H \hat{\otimes} K$ satisfies $\|x \otimes y\| = \|x\| \|y\|$ for $x \in H$ and $y \in K$. We use this to define a tensor product of two C*-algebras.

Let A be a C*-algebra with universal representation (H, π_A) and let B be a C*-algebra with universal representation (K, π_B). Then there is a a unique injective $*$-homomorphism $\phi : A \otimes B \to \mathcal{B}(H \hat{\otimes} K)$ such that $\phi(a \otimes b) = \pi_A(a) \otimes \pi_B(b)$. Thus we may define a C^*-norm on $A \otimes B$ by $\|c\|_{\min} = \|\phi(c)\|$ for $c \in A \otimes B$. Note that $\|a \otimes b\|_{\min} = \|a\|\|b\|$ for every $a \in A$ and $b \in B$.

Definition: The minimal (or spatial) tensor product of A and B is given by

$$A \otimes_{\min} B := \overline{A \otimes B}^{\|\cdot\|_{\min}}.$$

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